CLASSIFICATION OF DISTINCT MAXIMAL FLAG CODES OF A PRESCRIPED TYPE AND RELATED RESULTS

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ABSTRACT

CLASSIFICATION OF DISTINCT MAXIMAL FLAG CODES OF A PRESCRIPED TYPE AND RELATED RESULTS

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In this thesis, we aim to improve the current bounds for a certain type of code and not only find the number of distinct codes but also characterize them for some parameters.

Flag codes have applications in network coding and their algebraic and combinatorial structures have been an active research area in recent years, see, for example, [4, 23]. Characterization of maximal flag codes of a given type and distance over a given ambient space is a very difficult problem. In this work, we completely solve this problem for small parameters with the help of MAGMA [8]. In particular, we find new maximal flag codes as well. For a given type and distance of a flag code, the number of distinct flag codes are determined exactly for some parameters, and we give bounds for arbitrary ones. The concept of set flag codes are given nicely and it is shown that some of the bounds of [23] are not tight for all q.

Keywords: Graph Theory, Coding Theory, Flag Codes, Permanents

ÖZ

BELİRLİ BİR TİPTEKİ FARKLI MAKSİMAL BAYRAK KODLARININ SINIFLANDIRILMASI VE İLGİLİ SONUÇLAR

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Bu tezde, bayrak kodları için literatürde mevcut olan bazı sınırları geliştirmek ve belirli tipteki farklı kodların sayılarını vermekle birlikte bazı parametreler için onları karakterize etmeyi amaçlıyoruz.

Bayrak kodlarının ağ kodlamada önemli uygulamaları mevcuttur, ve onların cebirsel ve kombinatorik yapıları son yıllarda aktif bir araştırma alanıdır [4, 23]. Verilen ambiyant uzay üzerinde belirli bir tipe ve uzaklığa sahip maksimal bayrak kodların karakterizasyonu çok zor bir problemdir. Bu tezde, biz bu problemi küçük parametreler için MAGMA [8] kullanarak tamamen çözdük ve diğer tüm parametreler için çizge teorisindeki bazı yapılar üzerinden kombinatorik sonuçları da kullanarak birer alt sınır ve üst sınır belirledik. Verilen bir tipteki ve istenen bir uzaklığa sahip bayrak kodlarının sayısı için tam sonuçları bazı parametreler için belirledik, diğer değerler için de alt ve üst sınır verdik. Küme-bayrak-kodlarını tanımladık ve bu sayede [23]'da verilmiş olan bazı sınırların her q değeri için geçerli olmadığını gösterdik.

Anahtar Kelimeler: Çizge Teorisi, Kodlama Teorisi, Bayrak Kodları

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LIST OF ABBREVIATIONS

$\mathbb{F}_q$	Finite field with q elements
$\mathcal{P}_q(n)$	Subspace channel
$perA$	Permanent of a matrix A
$N(v)$	The set of adjacent vertices to a vertex v in a graph G
$rs(A)$	Row space that is spanned by the rows of a matrix A
$\mathcal{A}_{\mathbb{F}_q,n}(T;d)$	The number of maximum size of a flag code of type T, minimum distance d in $\mathbb{F}_q^n$ over $\mathbb{F}_q$
$\mathcal{M}_{\mathbb{F}_q,n}(T;d)$	The set of maximal flag codes of type T, minimum distance d in $\mathbb{F}_q^n$ over $\mathbb{F}_q$

CHAPTER 1

INTRODUCTION

Graphs are mathematical structures that can be used to model some theoretical problems. Euler introduced graph theory as a new branch in 1736 [13], and around a hundred years later Kirchoff made a significant contribution for electrical networks' analysis. It was Poincaré who gave the definition of incidence matrix very first time [26]. Since then there have been many works for the graph theory and its relation for other areas such as engineering, architecture, management and control and so on, some of the leading books for those relations are [6, 7, 15]. One of the most important areas is, non-surprisingly, communication.

Communication and data-transferring systems are greatly important in today's life. Reliable and fast communication is essential for all sorts of companies and individuals as well. Classical Coding and Information Theory works on a transmission from a sender $\mathcal{S}$ to an intended receiver $\mathcal{R}$.

$$\mathcal{S} \longrightarrow \mathcal{R}$$

Coding theory was born in the 1940s to solve the concern of the security of sending information via a noisy channel. Claude Shannon presented a way to calculate the maximum rate of data transmission with zero error happening in a channel with specific bandwidth and noise characteristics [28]. Following him, Richard Hamming studied error-correcting codes with information transmission rates more efficiently than simple repetition. He produced a code that four data bits followed by three bits for the check that admits not only the detection but also the correction of a single error. The fundamental of coding theory was given in [17].

Definition 1. *Let $\mathbf{A}$ be a set with a finitely many elements. A code $\mathcal{C}$ over $\mathbf{A}$ of length n is a subset of $\mathbf{A}^n$.*

In order to combat possible errors and/or erasures through the channel, the most common tool is the usage of linear block codes. Fix a finite field F (Mostly it is characteristic 2, as $F = \mathbb{F}_2$) and take a subspace U of K^n. Here U is considered as a code, and the elements $u \in U$ are being sent through the channel. If any change of entry of u occurs in the channel, then the receiver gets u' and the error can be characterized as

$$d_H := |\{1 \leq i \leq n : u_i \neq u_i'\}|.$$

This is the Hamming distance of u and u'. The bigger hamming distance they have, the more errors can be corrected by the nearest neighbor decoding algorithm. In fact, the error correcting capacity of the system is equal to $\lfloor \frac{\min d_H}{2} \rfloor$ since we could have at most one $u \in U$ with $d_h(u, u') = \lfloor \frac{\min d_H}{2} \rfloor$ and d_H satisfies the triangle inequality.

Because of the wide usage of the internet, the security of data transmission over a network is more and more important. A very natural question here could be how to execute the transfer of any data to more than one receiver, e.g. downloading or streaming anything. The most prominent answer to this question is to use *Network Coding*.

Network Coding was first introduced by [2] to attain a maximum information flow within a network. In [2], it has been shown that the usage of coding at the network nodes can be more useful than just routing the received inputs. After that, [22] provided an algebraic approach to coding for *random network coding.* They defined the *subspace channel*, given by $\mathcal{P}_q(n)$, as a discrete memoryless channel within the alphabets used to represent input and output. *Subspace codes* would be used to correct possible erasures or errors during the transmission. The usage of subspace codes was introduced by [22] as a sufficient communication channel in network coding from the sender node to the receiver node. This was the very first introduction of subspace codes to be used in communication [20].

In the concept of flag codes, first proposed by [24], the dimension of the transmitted subspace is fixed at each time, and it should contain the subspace(s) that is sent in previous time(s). In this way, the capacity for error correction improves. In [4], authors stated that when n is even and that if each transmission has the possible maximum distance, so does the flag, they call it optimum distance flag code. Their biggest motivation is some features of spreads as constant dimension codes, and they focused on the flag codes, which can be constructed from some spread. Also, they proved that having a planar spread at the k^{th} shot as a constant dimension code directs us to the best possible size for the optimum distance full flag codes when $n = 2k$.

Recently [4, 23] considered algebraic and combinatorial structures of flag codes. Let $\mathbb{F}_q$ be a finite field. Let $n \geq 2$ be an integer. Let $1 \leq s \leq n-1$ be an integer. Put $T = (t_1, \dots, t_s)$. By a flag of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$, we mean a chain of $\mathbb{F}_q$-linear subspaces

$$V_1 \subset V_2 \subset \dots \subset V_s \subset \mathbb{F}_q^n$$

such that

$$\dim_{\mathbb{F}_q}(V_1) = t_1,\ \dim_{\mathbb{F}_q}(V_2) = t_2, \dots,\ \dim_{\mathbb{F}_q}(V_s) = t_s.$$

There is a natural distance, which is the flag distance, between two flags of type-T, in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ (see Chapter 2 below), which generalizes the subspace distance (see Remark 1 below).

For some given integer d, it may be impossible to construct any flag code of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ with minimum flag distance d. We say d is admissible if there exists such a flag code (see Definition 6 below).

A fundamental problem is to determine the number of all maximal flag codes of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ with an admissible minimum flag distance d. This is a very difficult problem along with some interesting recent results, for example, [24, 4, 23].

A more difficult fundamental problem is to characterize all such flag codes with maximal cardinality. In this thesis, we solve this problem completely for small parameters and give bounds for general cases. Namely, our exact results for this problem include the following cases:

- $\mathbb{F}_q = \mathbb{F}_2,\ n = 3,\ T = (1, 2),\ d = 4,$
- $\mathbb{F}_q = \mathbb{F}_3,\ n = 3,\ T = (1, 2),\ d = 4,$

- $\mathbb{F}_q = \mathbb{F}_5,\ n = 3,\ T = (1, 2),\ d = 4,$
- $\mathbb{F}_q = \mathbb{F}_2,\ n = 4,\ T = (1, 2),\ d = 4.$

In particular, we find new explicit maximal flag codes (see Remarks 5, 6, 7 below).

With the help of combinatorics, even it is possible to solve some problems exactly or give some bounds. Hence, we use related materials to model our problem and give bounds for the number of distinct maximal flag codes.

This thesis is organized as follows. We fix the notation and give some preliminaries in Chapter 2. In Chapter 3, we model our problem in graph theory concept, and in Chapter 4, we solve the problem completely for the set-subset format. We also studied the equivalency for flag codes in Chapter 5 and our exact results and actual flag codes are given in Chapter 6 and Section 6.1. We present our results for $\mathbb{F}_q \in \{\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5\}$, $n = 3$, $T = (1, 2)$, and $d = 4$ along with $\mathbb{F}_q = \mathbb{F}_2$, $n = 4$, $T = (1, 2)$, and $d = 4$ in Chapter 6 and Section 6.1..

CHAPTER 2

PRELIMINARIES

In this chapter we give some necessary definitions and notions related to flag codes. Also, the knowledge of graph theory and its relation with coding theory is being set in this chapter.

Definition 2. *Let $n \geq 2$ be an integer. Let $1 \leq s \leq (n-1)$ be an integer. Let $1 \leq t_1 < t_2 < \cdots < t_s \leq (n-1)$. Put $T = (t_1, t_2, \ldots, t_s)$. By a **flag** of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$, we mean a chain of $\mathbb{F}_q$-linear subspaces*

$$V_1 \subset V_2 \subset \cdots \subset V_s \subset \mathbb{F}_q^n \tag{2.1}$$

such that

$$\dim_{\mathbb{F}_q}(V_1) = t_1,\ \dim_{\mathbb{F}_q}(V_2) = t_2, \ldots,\ \dim_{\mathbb{F}_q}(V_s) = t_s.$$

We also use the notation $[V_1 \subset V_2 \subset \cdots \subset V_s]$ to denote the flag in (2.1). Note that two flags $[V_1 \subset V_2 \subset \cdots \subset V_s]$ and $[U_1 \subset U_2 \subset \cdots \subset U_s]$ of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ are equal if and only if $V_1 = U_1, V_2 = U_2, \ldots$ and $V_s = U_s$.

Definition 3. *Let $n \geq 2$ be an integer. Let $1 \leq s \leq (n-1)$ be an integer. Let $1 \leq t_1 < t_2 < \cdots < t_s \leq (n-1)$. Put $T = (t_1, t_2, \ldots, t_s)$. By the **ambient space** of flags of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$, we mean the collection of $\mathcal{S}$ consisting of all flags of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ defined in Definition 2.*

Definition 4. *Let $n \geq 2$ be an integer. Let $1 \leq s \leq (n-1)$ be an integer. Let $1 \leq t_1 < t_2 < \cdots < t_s \leq (n-1)$. Put $T = (t_1, t_2, \ldots, t_s)$. By the flag distance $d_f([V_1 \subset V_2 \subset \cdots \subset V_s], [U_1 \subset U_2 \subset \cdots \subset U_s])$ of two flags $[V_1 \subset V_2 \subset \cdots \subset V_s]$*

and $[U_1 \subset U_2 \subset \cdots \subset U_s]$ *of type-T in* $\mathbb{F}_q^n$ *over* $\mathbb{F}_q$*, we mean*

$$d_f([V_1 \subset V_2 \subset \cdots \subset V_s], [U_1 \subset U_2 \subset \cdots \subset U_s]) = \sum_{i=1}^{s} d_S(U_i, V_i)$$

where

$$d_S(U_i, V_i) = \dim_{\mathbb{F}_q}(U_i) + \dim_{\mathbb{F}_q}(V_i) - 2\dim_{\mathbb{F}_q}(U_i \cap V_i)$$

for $1 \leq i \leq s$.

Remark 1. *If* $s = 1$ *and* $T = (t_1)$*, then the ambient space of flags of type-T in* $\mathbb{F}_q^n$ *over* $\mathbb{F}_q$ *is exactly the ambient space of constant* t_1*-dimensional* $\mathbb{F}_q$*-linear codes in* $\mathbb{F}_q^n$*, which is a Grassmannian. Moreover, in this case, the flag distance is the same as the subspace distance [19, 18, 20].*

Definition 5. *Let* $n \geq 2$ *be an integer. Let* $1 \leq s \leq (n-1)$ *be an integer. Let* $1 \leq t_1 < t_2 < \cdots < t_s \leq (n-1)$*. Put* $T = (t_1, t_2, \ldots, t_s)$*. By a flag code of type-T in* $\mathbb{F}_q^n$ *over* $\mathbb{F}_q$*, we mean a subset* $\mathcal{C} \subseteq \mathcal{S}$ *such that* $|\mathcal{C}| \geq 2$*, where* $\mathcal{S}$ *is the ambient space of flags of type-T in* $\mathbb{F}_q^n$ *over* $\mathbb{F}_q$*. Recall that* $\mathcal{S}$ *is defined in Definition 3. By the* ***minimum flag distance*** $d_f(\mathcal{C})$ *of the flag code* $\mathcal{C}$ *we mean*

$$\begin{aligned} d_f(\mathcal{C}) \; &= \min\{d_f([V_1 \subset V_2 \subset \cdots \subset V_s], [U_1 \subset U_2 \subset \cdots \subset U_s]) : \\ &\qquad [V_1 \subset V_2 \subset \cdots \subset V_s], [U_1 \subset U_2 \subset \cdots \subset U_s] \in \mathcal{C} \\ &\qquad \textit{and} \;\; [V_1 \subset V_2 \subset \cdots \subset V_s] \neq [U_1 \subset U_2 \subset \cdots \subset U_s]\}. \end{aligned}$$

Definition 6. *Let* $n \geq 2$ *be an integer. Let* $1 \leq s \leq (n-1)$ *be an integer. Let* $1 \leq t_1 < t_2 < \cdots < t_s \leq (n-1)$*. Put* $T = (t_1, t_2, \ldots, t_s)$*. Let* $d \geq 1$ *be an integer. We say that* d *is an* ***admissible*** *minimum distance for a flag code of type-T in* $\mathbb{F}_q^n$ *over* $\mathbb{F}_q$ *if there exists a flag code* $\mathcal{C}$ *of type-T in* $\mathbb{F}_q^n$ *over* $\mathbb{F}_q$ *such that* $d_f(\mathcal{C}) = d$*.*

Definition 7. *Let* $n \geq 2$ *be an integer. We say that* T *is the full type in* $\mathbb{F}_q^n$ *over* $\mathbb{F}_q$ *if* $s = (n-1)$ *and* $T = (1, 2, \ldots, n-1)$*. We say* $\mathcal{C}$ *is a* ***full*** *flag code in* $\mathbb{F}_q^n$ *over* $\mathbb{F}_q$ *if it is a flag code of the full type in* $\mathbb{F}_q^n$ *over* $\mathbb{F}_q$*.*

Remark 2. *Let* $n \geq 2$ *be an integer. Put* $T = (1, 2, \ldots, n-1)$*, i.e., the full type in* $\mathbb{F}_q^n$ *over* $\mathbb{F}_q$*. Let* $d \geq 1$ *be an integer. It is known that if* d *is an admissible minimum distance for a flag code of type-T in* $\mathbb{F}_q^n$ *over* $\mathbb{F}_q$ *(in the sense of Definition 6), then*

$$d_f(C) \leq \begin{cases} \frac{n^2}{2}, & n \textit{ is even}, \\ \frac{n^2-1}{2}, & n \textit{ is odd}. \end{cases} \tag{2.2}$$

We refer to [4] for proof.

Definition 8. *Let $n \geq 2$ be an integer. Let $1 \leq s \leq (n-1)$ be an integer. Let $1 \leq t_1 < t_2 < \cdots < t_s \leq (n-1)$. Put $T = (t_1, t_2, \ldots, t_s)$. Let $d \geq 1$ be an admissible minimum distance for a flag code of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$. Let $\mathcal{A}_{\mathbb{F}_q,n}(T; d)$ be the positive integer given by*

$$\mathcal{A}_{\mathbb{F}_q,n}(T; d) = \max\{|\mathcal{C}| : \mathcal{C} \text{ is a flag code of type-T in } \mathbb{F}_q^n \text{ over } \mathbb{F}_q \text{ with } d_f(\mathcal{C}) \geq d\}.$$

Remark 3. *Let $n \geq 2$ be an integer. Let $1 \leq s \leq (n-1)$ be an integer. Let $1 \leq t_1 < t_2 < \cdots < t_s \leq (n-1)$. Put $T = (t_1, t_2, \ldots, t_s)$. Let $d \geq 1$ be an admissible minimum distance for a flag code of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$. In general, it is a very difficult problem to determine $\mathcal{A}_{\mathbb{F}_q,n}(T; d)$. There are very interesting results for some parameters $\mathbb{F}_q, n, T, d$ in [23] and [4].*

The following two definitions are crucial for this paper.

Definition 9. *Let $n \geq 2$ be an integer. Let $1 \leq s \leq (n-1)$ be an integer. Let $1 \leq t_1 < t_2 < \cdots < t_s \leq (n-1)$. Put $T = (t_1, t_2, \ldots, t_s)$. Let $d \geq 1$ be an admissible minimum distance for a flag code of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$. Let $\mathcal{C}$ be a flag code of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ such that $d_f(\mathcal{C}) = d$. We say that $\mathcal{C}$ is a maximal flag code of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ with $d_f(\mathcal{C}) = d$ if*

$$|\mathcal{C}| = \mathcal{A}_{\mathbb{F}_q,n}(T; d).$$

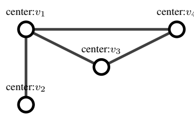
Definition 10. *Let $n \geq 2$ be an integer. Let $1 \leq s \leq (n-1)$ be an integer. Let $1 \leq t_1 < t_2 < \cdots < t_s \leq (n-1)$. Put $T = (t_1, t_2, \ldots, t_s)$. Let $d \geq 1$ be an admissible minimum distance for a flag code of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$. Let $\mathcal{M}_{\mathbb{F}_q,n}(T; d)$ be the set consisting of maximal flag codes $\mathcal{C}$ of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ with $d_f(\mathcal{C}) = d$. Recall that a maximal flag code $\mathcal{C}$ of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ with $d_f(\mathcal{C}) = d$ is defined in Definition 9.*

Remark 4. *In this paper, we observe that characterization of $\mathcal{M}_{\mathbb{F}_q,n}(T; d)$ given in Definition 10 is even more difficult than the determination of $\mathcal{A}_{\mathbb{F}_q,n}(T; d)$ given in Definition 8 for some parameters $\mathbb{F}_q, n, T, d$ (see Remark 3).*

In this work, we find a way to represent flag codes as graphs. If the type vector is a duple, we can use bipartite graphs. Moreover, for a general representation of the type vector, we can represent the nestedness with a partitioned hypergraph and the hyperedges on it. Hence, the graph-related definitions are given as follows.

Definition 11. *A graph G is defined as a pair $G = (V, E)$ where V is a finite set of elements called vertices, and E stands for the collection of unordered pairs of elements of V, called edges.*

Definition 12. *A graph $G = (V, E)$ is called simple if E has no repeated members, i.e., there exists one edge or no edge between every pair of vertices.*



Definition 13. *Let $G = (V, E)$ be a graph and an edge $e_i = \{u, v\}$, we say*

1. *the vertices u and v are the endpoints of e_i,*
2. *u and v are adjacent vertices and write $u \sim v$, otherwise $u \nsim v$.*
3. *it is said that the edge e_i is incident to u and v.*

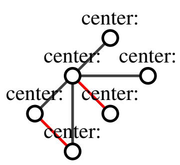
Definition 14. *The number of edges that are incident to any specific vertex $v \in V$, defines its degree.*

Definition 15. *The set of all vertices which are adjacent to a specific vertex $v \in V$ is called neighborhood of v and denoted as $N(v)$. Namely,*

$$N(v) = \{u \in V : \{u, v\} \in E\}.$$

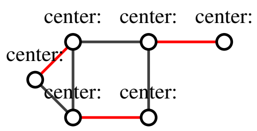
So, the cardinality of $N(v)$ gives us the degree of v.

Definition 16. *A subset $M \subset E$ is called an independent set or matching if none of its members is incident to the same vertex.*



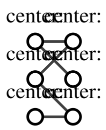
Definition 17. *A matching in a graph that cannot be extended is called as maximal matching and the matching with a maximum possible cardinality is called maximum matching.*

Definition 18. *A perfect matching of a graph is a matching (i.e., an independent edge set) in which every vertex of the graph is incident to exactly one edge of the matching.*



Definition 19. *In a graph $G(V, E)$, if all the vertices have the same degree, say k, the graph itself is called k-regular.*

Definition 20. *For a graph $G(V, E)$, if we can partition the vertex set V into two disjoint sets, say U and U', such that every member of E is incident to both sides' members, i.e., $\forall e_i \in E, e_i = \{u_i', u_i\}$ where $u_i \in U$ and $u_i' \in U'$, then this graph is called as a bipartite graph.*



Definition 21. *Consider a graph $G = (V, E)$. An $|V| \times |V|$ matrix A is called adjacency matrix if its elements $a_{ij} = 1$ if vertices v_i and v_j are adjacent and $a_{ij} = 0$ otherwise, where $v_i, v_j \in V$. A is a symmetric matrix and the sum of each row element gives the degree of corresponding vertex.*

Definition 22. *A hypergraph is a generalization of a graph and denoted as $H(V, E)$ on a finite set of $V = \{V_1, \ldots, V_n\}$ is defined as a family of p hyperedges $E = \{e_1, \ldots, e_p\}$ where each hyperedge is a non-empty subset of V.*

Definition 23. *Let $k \in \mathbb{Z}^+$, for a hypergraph if all hyperedges have the same cardinality k, then this hypergraph is called as k-uniform hypergraph.*

For example every simple graph is a *2-uniform hypergraph.*

For an arbitrary length n, type-$(\ell, n-\ell)$, minimum distance $2(\ell + (n-\ell))$ such that $1 \leq \ell < \frac{n}{2}$, the size of the flag code here is equivalent to the size of ℓ-dimensional Grassmannian of $\mathbb{F}_q^n$. (It is also equal to the $(n-\ell)$dimensional Grassmannian).

Consider a bipartite graph whose vertices on one side are composed of V_{ℓ_i}'s and the vertices of the other side are representing $V_{(n-\ell)_j}$ and there is an edge between any two of them if and only if $V_{\ell_i} \subset V_{(n-\ell)_j}$. This graph is simple, each side has $\begin{bmatrix} n \\ \ell \end{bmatrix}_q = \begin{bmatrix} n \\ n-\ell \end{bmatrix}_q$ vertices. Also, this graph is $\begin{bmatrix} n-\ell \\ \ell \end{bmatrix}_q$-regular. This will be called as an "$(\ell, n-\ell)$ *Grassmannian Correspondence Graph.*" For simplicity we will denote $\alpha := \begin{bmatrix} n \\ \ell \end{bmatrix}_q$ and $\beta := \begin{bmatrix} n-\ell \\ \ell \end{bmatrix}_q$.

The existence of such perfect matching is secured by the result of [16]. Asking how many flag codes of type-$(\ell, n-\ell)$ in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ and minimum distance $2(\ell + (n-\ell))$ exist is the same question with asking how many complete point-line (plane-hyperplane) correspondence exist. In this work, we find the exact result for this question for some small parameters. However, type-$(\ell, n-\ell)$ and in an n-dimensional Vector Space, finding the number of all maximal flag codes does not seem to be an easy question. We obtain upper and lower bounds for type-$(\ell, n-\ell)$ for an arbitrary length n as a consequence of [3, 9, 25, 27].

Permanent of an $n \times n$ matrix $A = [a_{ij}]_{n \times n}$ is defined by

$$perA = \sum \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of $\{1, 2, \ldots, n\}$. The permanent of a $(0,1)$-matrix can be interpreted as the number of perfect matchings in a bipartite graph, [5].

A permanent of a bipartite graph whose partitions represent members of ℓ-dimensional Grassmannian and $(n-\ell)$-dimensional Grassmannian is the number of perfect matchings occur in the graph and that is exactly an illustration of maximal flag codes of type-$(\ell, n-\ell)$ in $\mathbb{F}_q^n$ over $\mathbb{F}_q$.

In order to generalize adjacency matrices to higher orders, we can use multi-dimensional arrays called tensors. Let $\mathbb{T}$ be an r-dimensional tensor that has size $n_1 \times \cdots \times n_r$.

Each element of $\mathbb{T}$ is shown with $\mathbb{T}_{i_1,\ldots,i_r}$ where $i_k \in \{1,\ldots,n_j\}$. Here we need another material called *marginal*.

Definition 24. *Let $\mathbb{T}$ be an r-dimensional tensor. A marginal is defined as an $(r-1)-$section of an r-dimensional tensor which is derived by fixing one of the indices of $\mathbb{T}$.*

In the literature, especially the work [11], tells us that finding a maximum cardinality matching in an *r-partite*, *r-uniform* hypergraph for $r \geq 3$ is NP-Complete. The case corresponding to triple-type vector maximal flag codes is called as MAX-3DM problem [21]. To define the exact number there have been studies in the literature and the best-known approximation belongs to [10]. Here even $r = 3$ case has not been determined clearly, yet. However, the bounds of the number of maximal flag codes for an arbitrary type can be extended for hypermatrices deriving nicely from the incidence relations of hyperedges in the corresponding Grassmannian Correspondence Hypergraph.

CHAPTER 3

BOUNDS ON THE NUMBER OF MAXIMAL FLAG CODES FOR CERTAIN FORMS OF THE TYPE VECTOR

The upper bound for the most general case we achieved is given in the first Theorem. This is derived by modeling all of the following works to the extension of results found for bipartite simple graphs to r-partite, r-uniform hypergraphs.

Theorem 1. *If we are given a type vector $T^* := (t_1, \dots, t_r)$ and tasked to build a flag code with type-T^* in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ the followings are true:*

1. *Minimum admissible flag distance, d^* is equal to $2r$,*

2. $\mathcal{A}_{\mathbb{F}_q,n}(T^*; d^*) = \min\{|\mathcal{G}_q(t_1, n)|, \dots, |\mathcal{G}_q(t_r, n)|\}$,

3. $|\mathcal{M}_{\mathbb{F}_q,n}(T^*; d^*)| \leq \prod_{i}^{N} (\gamma_i!)^{\gamma_i}$.

where γ_i is the sum of marginals of the corresponding tensor and N is the total number of vertices of the Grassmannian Correspondence Hypergraph.

Proof. 1. It is a direct result of the subspace distance for distinct elements of a Grassmannian.

2. When we model our flag to r-partite, r-uniform hypergraphs, we realize that each hyperedge occurs by the relation of the nested subspaces of $\mathbb{F}_q^n$ over $\mathbb{F}_q$. Therefore, the size of the maximum matching is limited by the minimum size of the parts.

3. A maximal flag code of type-$(t_1, \ldots, t_r)$ with an admissible distance can be represented with an *r-partite* and *r-uniform* hypergraph where the vertex set can be written as a union $V = \cup_{i=1}^{r} V_i$ with disjoint V_i's and each of hyperedges is incident to a single vertex from each V_i. The existence of a matching in an r-dimensional hypergraph was given by [1] which provides equivalency of Hall's Theorem for r-partite hypergraphs.

   Therefore, an r-partite, r-uniform hypergraph $H = (V_1 \cup \cdots \cup V_r, E)$ can be represented as an r-dimensional tensor. This completes the association of each vertex class to a tensor dimension. Let $|V_i| = n_i$ for $1 \leq i \leq r$, the tensor $\mathbb{T} \in \{0,1\}^{n_1 \times \cdots \times n_r}$ has a nonzero element $\mathbb{T}_{n_1,\ldots,n_r}$ if and only if $e_0 = \{v_1, \ldots, v_r\} \in E$, where $v_i \in V_i$ for $1 \leq i \leq r$. Then, $\mathbb{T}$ is called *the adjacency tensor of* H.

   If we assign γ_i to the summation of each marginal of the adjacency tensor $\mathbb{T}$, then following the works done by [3] and references therein, we can reach the upper bound given.

□

Theorem 2. *Let $\mathbb{F}_q$ be a finite field for an arbitrary prime q and $\mathbb{F}_q^n$ is an n-dimensional Vector Space over $\mathbb{F}_q$. The set $\mathcal{M}_{\mathbb{F}_q,n}((\ell, n-\ell); 4\ell)$ of maximal flag codes of type-$(\ell, n-\ell)$ in $\mathbb{F}_q^n$ is bounded as*

$$(\frac{\beta}{\alpha})^{\alpha}.\alpha! \leq |\mathcal{M}_{\mathbb{F}_q,n}((\ell, n-\ell); 4\ell)| \leq \prod_{1}^{\alpha} (\beta!)^{\frac{1}{\beta}}. \tag{3.1}$$

Recall that $\alpha = \begin{bmatrix} n \\ \ell \end{bmatrix}_q$ and $\beta = \begin{bmatrix} n-\ell \\ \ell \end{bmatrix}_q$.

Proof. Consider G as the $(\ell, n-\ell)$ Grassmannian Correspondence Graph. Here, the total number of the edges in G is $\alpha.\ \beta$.

Having type-$(\ell, n-\ell)$ flag code with minimum distance 4ℓ sets a perfect matching in the graph described above. Remember that $\ell < \frac{n}{2}$ and $d_S = 2\ell - 2\dim(U_i \cap V_i)$ for constant dimension codes. It is a perfect matching because in each code of constant dimension, subspaces are used once.

The adjacency matrix of this graph is a $\begin{bmatrix} n \\ \ell \end{bmatrix}_q \times \begin{bmatrix} n \\ \ell \end{bmatrix}_q$ dimensioned matrix whose entries are either 0 or 1 according to the corresponding vertices' relations. If the vertex v_i is connected to the vertex v_j with an edge, then the entry a_{ij} of the matrix A is 1, otherwise 0.

After all these settlements, counting distinct maximal flag codes of type- $(\ell, n-\ell)$ with minimum distance 4ℓ is just the same as counting the number of distinct perfect matchings in a bipartite graph and this is an NP-complete problem.

For any vector space and any ℓ, the number of distinct maximal flag codes can be interpreted as the number of permanents of a $(0,1)$-matrix which is also the number of perfect matchings in a balanced β-regular bipartite graph.

For the upper bound of (3.2) we use the number of spanning 2-regular subgraphs of G, namely H which consist of even cycles such that H is being counted 2^c times, where c is the number of such cycles within more than 2 vertices. In fact, each of the perfect matching pairs P_1, P_2 is a copy of H which is a 2-regular subgraph of G. For the cycles of length $t_{>2}$, we have two possibilities of its origin, either P_1 or P_2.

On the other hand, the number of 2-regular subgraphs of G is equal to the permanent of the adjacency matrix. Here cycles with odd lengths and length 2 are allowed. Such subgraphs are counted 2^c, where c is the number of length $t_{>2}$ cycles. Hence the square root of the number of permanents for the adjacency matrix is the limit of the number of perfect matchings. Therefore, the desired limit is achieved by the work of Bregman-Minc, where β is the cardinality of each of α vertices in G.

For the lower bound, we will use the results of [12] and [14]. Here our $(\ell, n-\ell)$ Grassmannian Correspondence Graph has an $\alpha \times \alpha$ adjacency matrix that is β-regular. The number of perfect matchings here bounded by $(\frac{\beta}{\alpha})^{\alpha}.\ \alpha!$. This bound has been carefully selected among the results in the literature as the best match to the number we seek as β will be the sum of each row.

$\square$

The lower bound in 3.2 gives better results for some cases in Table 3.1. For higher characteristics, the bound given below gives better results as seen in the table.

Theorem 3. *Let $\mathbb{F}_q$ be a finite field for an arbitrary prime q and $\mathbb{F}_q^n$ is an n-dimensional Vector Space over $\mathbb{F}_q$. The set $\mathcal{M}_{\mathbb{F}_q,n}((\ell, n-\ell); 4\ell)$ of maximal flag codes of type-$(\ell, n-\ell)$ in $\mathbb{F}_q^n$ is bounded as*

$$(\frac{(\beta-1)^{\beta-1}}{\beta^{\beta-2}})^{\alpha} \leq |\mathcal{M}_{\mathbb{F}_q,n}((\ell, n-\ell); 4\ell)|. \tag{3.2}$$

In Table 3.1 we compare the upper bounds and lower bounds with the exact results that we know for small parameters.

For an arbitrary duble type vector apart from $(\ell, n-\ell)$, the Grassmannian Correspondence Graph will be an unbalanced bipartite graph. If the type vector in the form of (t_1, t_2) for any $1 \leq t_1 < t_2 \leq (n-1)$, then the volumes of the parts of our graph will be $\begin{bmatrix} n \\ t_1 \end{bmatrix}_q$ and $\begin{bmatrix} n \\ t_2 \end{bmatrix}_q$. Moreover, each of the left-hand side vertices is connected to the same number of right-hand side ones. The number of maximal flag codes of type-(t_1, t_2) will correspond to the number of maximum matchings here.

Table 3.1: Maximal Flag Codes of Some Certain Types

$Length(n)$	type	q	Exact Results	Upper Bounds	Lower Bounds by 2	Lower Bounds by 3
3	$(1,2)$	2	24	65	13	7,49
3	$(1,2)$	3	3 852	30 597	1379	899
3	$(1,2)$	4	18 534 400	540 208 276	4 169 738	3 450 873
3	$(1,2)$	5	4 598 378 639 550	579 274 705 236 857	638 995 826 718	707 286 916 498
3	$(1,2)$	7	*	$6,52 \times 10^{32}$	$9,98 \times 10^{27}$	$2,17 \times 10^{28}$
3	$(1,2)$	8	*	$1,25 \times 10^{45}$	$1,94 \times 10^{39}$	$6,11 \times 10^{39}$
3	$(1,2)$	9	*	$4,94 \times 10^{59}$	$7,22 \times 10^{52}$	$3,35 \times 10^{53}$
3	$(1,2)$	11	*	$1,61 \times 10^{96}$	$1,7 \times 10^{87}$	$1,76 \times 10^{88}$
3	$(1,2)$	13	*	$1,02 \times 10^{143}$	$6,25 \times 10^{131}$	$1,49 \times 10^{132}$
4	$(1,3)$	2	24 601 472	85 857 793	14 177 555	4 479 809
4	$(1,3)$	3	*	$1,37 \times 10^{30}$	$2,43 \times 10^{28}$	$7,44 \times 10^{27}$
4	$(1,3)$	4	*	$5,91 \times 10^{79}$	$6,85 \times 10^{76}$	$2,33 \times 10^{76}$
4	$(1,3)$	5	*	$4,67 \times 10^{170}$	$2,5 \times 10^{166}$	$1,02 \times 10^{166}$
4	$(1,3)$	7	*	$3,97 \times 10^{537}$	$2,15 \times 10^{530}$	$1,46 \times 10^{530}$
4	$(1,3)$	8	*	$4,46 \times 10^{846}$	$5,81 \times 10^{837}$	$5,36 \times 10^{837}$
4	$(1,3)$	9	*	$5,23 \times 10^{1262}$	$1,41 \times 10^{1252}$	$1,8 \times 10^{1252}$
5	$(1,4)$	2	*	$1,09 \times 10^{25}$	$1,39 \times 10^{24}$	$2,85 \times 10^{23}$
5	$(1,4)$	3	*	$1,44 \times 10^{146}$	$1,09 \times 10^{144}$	$1,75 \times 10^{143}$
5	$(1,4)$	4	*	$2,04 \times 10^{515}$	$3,18 \times 10^{511}$	$5,15 \times 10^{510}$
5	$(2,3)$	2	*	$9,6 \times 10^{81}$	$1,48 \times 10^{65}$	$5,36 \times 10^{68}$
6	$(1,5)$	2	*	$8,4 \times 10^{68}$	$7,85 \times 10^{67}$	$1,1 \times 10^{67}$
6	$(2,4)$	2	*	$9,4 \times 10^{725}$	$3,21 \times 10^{706}$	$4,83 \times 10^{708}$
7	$(1,6)$	2	*	$9,55 \times 10^{175}$	$6,49 \times 10^{174}$	$6,32 \times 10^{173}$
7	$(2,5)$	2	*	$1,16 \times 10^{4709}$	$2,91 \times 10^{4685}$	**
7	$(3,4)$	2	*	$6,83 \times 10^{1126}$	$5,99 \times 10^{1036}$	**
8	$(1,7)$	2	*	$5,07 \times 10^{475}$	$1,25 \times 10^{474}$	$1,45 \times 10^{426}$

* unknown in the literature.

** computationally hard to calculate.

CHAPTER 4

MAXIMAL SET FLAG CODES

Instead of an n-dimensional vector space V and its subspaces V_i, we consider a set S of cardinality n and its subsets.

The subspace distance corresponds to the subset distance given by

$$d_{set_f}(S_i, S_j) = \#(S_i \cup S_j) - \#(S_i \cap S_j).$$

For distinct subsets with same number of elements, $d_{set_f} = 2$. Therefore, in the set flag code concept: if our type vector has r elements, then the lower bound for the minimum admissible distance for the set flag code is $2r$.

The exact results of 4.1 can be detailed as follows:

Case 1: $n = 3$, type-$(1, 2)$, $d_{set_f} = 4$. Then, say $A = \{1, 2, 3\}$ and so the subsets are listed as:

$$\begin{aligned} V_1 &= \{1\}, & W_1 &= \{1, 2\}, \\ V_2 &= \{2\}, & W_2 &= \{1, 3\}, \\ V_3 &= \{3\}, & W_3 &= \{2, 3\}. \end{aligned}$$

Then, the possible distinct set flag codes are:

$$\begin{aligned} \mathcal{F}_1 &= \{[V_1 \subset W_1], [V_2 \subset W_2], [V_3 \subset W_3]\}, \\ \mathcal{F}_2 &= \{[V_1 \subset W_2], [V_2 \subset W_1], [V_3 \subset W_3]\}. \end{aligned}$$

Case 2: $n = 4$, type-$(1,3)$, $d_{set_f} = 4$. Then, say $A = \{1,2,3,4\}$ and so the subsets are listed as:

$$\begin{aligned} V_1 &= \{1\}, & W_1 &= \{1,2,3\}, \\ V_2 &= \{2\}, & W_2 &= \{1,2,4\}, \\ V_3 &= \{3\}, & W_3 &= \{1,3,4\}, \\ V_4 &= \{4\}, & W_4 &= \{2,3,4\}. \end{aligned}$$

Then, the possible distinct set flag codes are:

$$\begin{aligned} \mathcal{F}_1 &= \{[V_1 \subset W_1], [V_2 \subset W_2], [V_3 \subset W_3], [V_4 \subset W_4]\}, \\ \mathcal{F}_2 &= \{[V_1 \subset W_1], [V_2 \subset W_2], [V_3 \subset W_4], [V_4 \subset W_3]\}, \\ \mathcal{F}_3 &= \{[V_1 \subset W_1], [V_2 \subset W_4], [V_3 \subset W_3], [V_4 \subset W_2]\}, \\ \mathcal{F}_4 &= \{[V_1 \subset W_2], [V_2 \subset W_1], [V_3 \subset W_3], [V_4 \subset W_4]\}, \\ \mathcal{F}_5 &= \{[V_1 \subset W_2], [V_2 \subset W_1], [V_3 \subset W_4], [V_4 \subset W_3]\}, \\ \mathcal{F}_6 &= \{[V_1 \subset W_2], [V_2 \subset W_4], [V_3 \subset W_1], [V_4 \subset W_3]\}, \\ \mathcal{F}_7 &= \{[V_1 \subset W_3], [V_2 \subset W_1], [V_3 \subset W_4], [V_4 \subset W_2]\}, \\ \mathcal{F}_8 &= \{[V_1 \subset W_3], [V_2 \subset W_2], [V_3 \subset W_1], [V_4 \subset W_4]\}, \\ \mathcal{F}_9 &= \{[V_1 \subset W_3], [V_2 \subset W_4], [V_3 \subset W_1], [V_4 \subset W_2]\}. \end{aligned}$$

Case 3: $n = 4$, type-$(1,2)$, $d_{set_f} = 4$. Then, say $A = \{1,2,3,4\}$ and so the subsets are listed as:

$$\begin{aligned} & & W_1 &= \{1,2\}, \\ V_1 &= \{1\}, & W_2 &= \{1,3\}, \\ V_2 &= \{2\}, & W_3 &= \{1,4\}, \\ V_3 &= \{3\}, & W_4 &= \{2,3\}, \\ V_4 &= \{4\}, & W_5 &= \{2,4\}, \\ & & W_6 &= \{3,4\}. \end{aligned}$$

Then, the possible distinct set flag codes are:

$$
\begin{aligned}
\mathcal{F}_1 &= \{[V_1 \subset W_1], [V_2 \subset W_4], [V_3 \subset W_2], [V_4 \subset W_3]\},\\
\mathcal{F}_2 &= \{[V_1 \subset W_1], [V_2 \subset W_4], [V_3 \subset W_2], [V_4 \subset W_5]\},\\
\mathcal{F}_3 &= \{[V_1 \subset W_1], [V_2 \subset W_4], [V_3 \subset W_2], [V_4 \subset W_6]\},\\
\mathcal{F}_4 &= \{[V_1 \subset W_1], [V_2 \subset W_4], [V_3 \subset W_6], [V_4 \subset W_3]\},\\
\mathcal{F}_5 &= \{[V_1 \subset W_1], [V_2 \subset W_4], [V_3 \subset W_6], [V_4 \subset W_5]\},\\
\mathcal{F}_6 &= \{[V_1 \subset W_1], [V_2 \subset W_5], [V_3 \subset W_2], [V_4 \subset W_3]\},\\
\mathcal{F}_7 &= \{[V_1 \subset W_1], [V_2 \subset W_5], [V_3 \subset W_2], [V_4 \subset W_6]\},\\
\mathcal{F}_8 &= \{[V_1 \subset W_1], [V_2 \subset W_5], [V_3 \subset W_4], [V_4 \subset W_3]\},\\
\mathcal{F}_9 &= \{[V_1 \subset W_1], [V_2 \subset W_5], [V_3 \subset W_4], [V_4 \subset W_6]\},\\
\mathcal{F}_{10} &= \{[V_1 \subset W_1], [V_2 \subset W_5], [V_3 \subset W_6], [V_4 \subset W_3]\},\\
\mathcal{F}_{11} &= \{[V_1 \subset W_2], [V_2 \subset W_1], [V_3 \subset W_4], [V_4 \subset W_3]\},\\
\mathcal{F}_{12} &= \{[V_1 \subset W_2], [V_2 \subset W_1], [V_3 \subset W_4], [V_4 \subset W_5]\},\\
\mathcal{F}_{13} &= \{[V_1 \subset W_2], [V_2 \subset W_1], [V_3 \subset W_4], [V_4 \subset W_6]\},\\
\mathcal{F}_{14} &= \{[V_1 \subset W_2], [V_2 \subset W_1], [V_3 \subset W_6], [V_4 \subset W_3]\},\\
\mathcal{F}_{15} &= \{[V_1 \subset W_2], [V_2 \subset W_1], [V_3 \subset W_6], [V_4 \subset W_5]\},\\
\mathcal{F}_{16} &= \{[V_1 \subset W_2], [V_2 \subset W_4], [V_3 \subset W_6], [V_4 \subset W_3]\},\\
\mathcal{F}_{17} &= \{[V_1 \subset W_2], [V_2 \subset W_4], [V_3 \subset W_6], [V_4 \subset W_5]\},\\
\mathcal{F}_{18} &= \{[V_1 \subset W_2], [V_2 \subset W_5], [V_3 \subset W_4], [V_4 \subset W_3]\},\\
\mathcal{F}_{19} &= \{[V_1 \subset W_2], [V_2 \subset W_5], [V_3 \subset W_4], [V_4 \subset W_6]\},\\
\mathcal{F}_{20} &= \{[V_1 \subset W_2], [V_2 \subset W_5], [V_3 \subset W_6], [V_4 \subset W_3]\},\\
\mathcal{F}_{21} &= \{[V_1 \subset W_3], [V_2 \subset W_1], [V_3 \subset W_2], [V_4 \subset W_5]\},\\
\mathcal{F}_{22} &= \{[V_1 \subset W_3], [V_2 \subset W_1], [V_3 \subset W_2], [V_4 \subset W_6]\},\\
\mathcal{F}_{23} &= \{[V_1 \subset W_3], [V_2 \subset W_1], [V_3 \subset W_4], [V_4 \subset W_5]\},\\
\mathcal{F}_{24} &= \{[V_1 \subset W_3], [V_2 \subset W_1], [V_3 \subset W_4], [V_4 \subset W_6]\},\\
\mathcal{F}_{25} &= \{[V_1 \subset W_3], [V_2 \subset W_1], [V_3 \subset W_6], [V_4 \subset W_5]\},\\
\mathcal{F}_{26} &= \{[V_1 \subset W_3], [V_2 \subset W_4], [V_3 \subset W_2], [V_4 \subset W_5]\},\\
\mathcal{F}_{27} &= \{[V_1 \subset W_3], [V_2 \subset W_4], [V_3 \subset W_2], [V_4 \subset W_6]\},\\
\mathcal{F}_{28} &= \{[V_1 \subset W_3], [V_2 \subset W_4], [V_3 \subset W_6], [V_4 \subset W_5]\},\\
\mathcal{F}_{29} &= \{[V_1 \subset W_3], [V_2 \subset W_5], [V_3 \subset W_2], [V_4 \subset W_6]\},\\
\mathcal{F}_{30} &= \{[V_1 \subset W_3], [V_2 \subset W_5], [V_3 \subset W_4], [V_4 \subset W_6]\}.
\end{aligned}
$$

Case 4: $n = 4$, type-$(2, 3)$, $d_{set_f} = 4$. Then, say $A = \{1, 2, 3, 4\}$ and so the subsets are listed as:

$$
\begin{aligned}
W_1 &= \{1, 2\}, & & \\
W_2 &= \{1, 3\}, & Y_1 &= \{1, 2, 3\},\\
W_3 &= \{1, 4\}, & Y_2 &= \{1, 2, 4\},\\
W_4 &= \{2, 3\}, & Y_3 &= \{1, 3, 4\},\\
W_5 &= \{2, 4\}, & Y_4 &= \{2, 3, 4\}.\\
W_6 &= \{3, 4\}, & &
\end{aligned}
$$

Then, the possible distinct set flag codes are:

$$
\begin{aligned}
\mathcal{F}_1 &= \{[W_1 \subset Y_1], [W_3 \subset Y_2], [W_2 \subset Y_3], [W_4 \subset Y_4]\},\\
\mathcal{F}_2 &= \{[W_1 \subset Y_1], [W_3 \subset Y_2], [W_2 \subset Y_3], [W_5 \subset Y_4]\},\\
\mathcal{F}_3 &= \{[W_1 \subset Y_1], [W_3 \subset Y_2], [W_2 \subset Y_3], [W_6 \subset Y_4]\},\\
\mathcal{F}_4 &= \{[W_1 \subset Y_1], [W_3 \subset Y_2], [W_6 \subset Y_3], [W_4 \subset Y_4]\},\\
\mathcal{F}_5 &= \{[W_1 \subset Y_1], [W_3 \subset Y_2], [W_6 \subset Y_3], [W_5 \subset Y_4]\},\\
\mathcal{F}_6 &= \{[W_1 \subset Y_1], [W_5 \subset Y_2], [W_2 \subset Y_3], [W_4 \subset Y_4]\},\\
\mathcal{F}_7 &= \{[W_1 \subset Y_1], [W_5 \subset Y_2], [W_2 \subset Y_3], [W_6 \subset Y_4]\},\\
\mathcal{F}_8 &= \{[W_1 \subset Y_1], [W_5 \subset Y_2], [W_3 \subset Y_3], [W_4 \subset Y_4]\},\\
\mathcal{F}_9 &= \{[W_1 \subset Y_1], [W_5 \subset Y_2], [W_3 \subset Y_3], [W_6 \subset Y_4]\},\\
\mathcal{F}_{10} &= \{[W_1 \subset Y_1], [W_5 \subset Y_2], [W_6 \subset Y_3], [W_4 \subset Y_4]\},\\
\mathcal{F}_{11} &= \{[W_2 \subset Y_1], [W_1 \subset Y_2], [W_3 \subset Y_3], [W_4 \subset Y_4]\},\\
\mathcal{F}_{12} &= \{[W_2 \subset Y_1], [W_1 \subset Y_2], [W_3 \subset Y_3], [W_5 \subset Y_4]\},\\
\mathcal{F}_{13} &= \{[W_2 \subset Y_1], [W_1 \subset Y_2], [W_3 \subset Y_3], [W_6 \subset Y_4]\},\\
\mathcal{F}_{14} &= \{[W_2 \subset Y_1], [W_1 \subset Y_2], [W_6 \subset Y_3], [W_4 \subset Y_4]\},\\
\mathcal{F}_{15} &= \{[W_2 \subset Y_1], [W_1 \subset Y_2], [W_6 \subset Y_3], [W_5 \subset Y_4]\},\\
\mathcal{F}_{16} &= \{[W_2 \subset Y_1], [W_3 \subset Y_2], [W_6 \subset Y_3], [W_4 \subset Y_4]\},\\
\mathcal{F}_{17} &= \{[W_2 \subset Y_1], [W_3 \subset Y_2], [W_6 \subset Y_3], [W_5 \subset Y_4]\},\\
\mathcal{F}_{18} &= \{[W_2 \subset Y_1], [W_5 \subset Y_2], [W_3 \subset Y_3], [W_4 \subset Y_4]\},\\
\mathcal{F}_{19} &= \{[W_2 \subset Y_1], [W_5 \subset Y_2], [W_3 \subset Y_3], [W_6 \subset Y_4]\},\\
\mathcal{F}_{20} &= \{[W_2 \subset Y_1], [W_5 \subset Y_2], [W_6 \subset Y_3], [W_4 \subset Y_4]\},\\
\mathcal{F}_{21} &= \{[W_4 \subset Y_1], [W_1 \subset Y_2], [W_2 \subset Y_3], [W_5 \subset Y_4]\},\\
\mathcal{F}_{22} &= \{[W_4 \subset Y_1], [W_1 \subset Y_2], [W_2 \subset Y_3], [W_6 \subset Y_4]\},\\
\mathcal{F}_{23} &= \{[W_4 \subset Y_1], [W_1 \subset Y_2], [W_3 \subset Y_3], [W_5 \subset Y_4]\},\\
\mathcal{F}_{24} &= \{[W_4 \subset Y_1], [W_1 \subset Y_2], [W_3 \subset Y_3], [W_6 \subset Y_4]\},\\
\mathcal{F}_{25} &= \{[W_4 \subset Y_1], [W_1 \subset Y_2], [W_6 \subset Y_3], [W_5 \subset Y_4]\},\\
\mathcal{F}_{26} &= \{[W_4 \subset Y_1], [W_3 \subset Y_2], [W_2 \subset Y_3], [W_5 \subset Y_4]\},\\
\mathcal{F}_{27} &= \{[W_4 \subset Y_1], [W_3 \subset Y_2], [W_2 \subset Y_3], [W_6 \subset Y_4]\},\\
\mathcal{F}_{28} &= \{[W_4 \subset Y_1], [W_3 \subset Y_2], [W_6 \subset Y_3], [W_5 \subset Y_4]\},\\
\mathcal{F}_{29} &= \{[W_4 \subset Y_1], [W_5 \subset Y_2], [W_2 \subset Y_3], [W_6 \subset Y_4]\},\\
\mathcal{F}_{30} &= \{[W_4 \subset Y_1], [W_5 \subset Y_2], [W_3 \subset Y_3], [W_6 \subset Y_4]\}.
\end{aligned}
$$

Case 5: $n = 4$, type-$(1, 2, 3)$, $d_{set_f} = 6$. Then, say $A = \{1, 2, 3, 4\}$ and so the subsets are listed as:

$$
\begin{aligned}
& & W_1 &= \{1,2\}, & & \\
V_1 &= \{1\}, & W_2 &= \{1,3\}, & Y_1 &= \{1,2,3\}, \\
V_2 &= \{2\}, & W_3 &= \{1,4\}, & Y_2 &= \{1,2,4\}, \\
V_3 &= \{3\}, & W_4 &= \{2,3\}, & Y_3 &= \{1,3,4\}, \\
V_4 &= \{4\}, & W_5 &= \{2,4\}, & Y_4 &= \{2,3,4\}. \\
& & W_6 &= \{3,4\}, & &
\end{aligned}
$$

Then, the possible distinct full set flag codes are:

$$
\begin{aligned}
\mathcal{F}_1 &= \{[V_1 \subset W_1 \subset Y_1], [V_2 \subset W_4 \subset Y_2], [V_3 \subset W_2 \subset Y_3], [V_4 \subset W_3 \subset Y_4]\}, \\
\mathcal{F}_2 &= \{[V_1 \subset W_1 \subset Y_2], [V_2 \subset W_4 \subset Y_3], [V_3 \subset W_2 \subset Y_1], [V_4 \subset W_3 \subset Y_4]\}, \\
\mathcal{F}_3 &= \{[V_1 \subset W_1 \subset Y_1], [V_2 \subset W_4 \subset Y_2], [V_3 \subset W_2 \subset Y_3], [V_4 \subset W_5 \subset Y_4]\}, \\
\mathcal{F}_4 &= \{[V_1 \subset W_1 \subset Y_2], [V_2 \subset W_4 \subset Y_3], [V_3 \subset W_2 \subset Y_1], [V_4 \subset W_5 \subset Y_4]\}, \\
\mathcal{F}_5 &= \{[V_1 \subset W_1 \subset Y_1], [V_2 \subset W_4 \subset Y_2], [V_3 \subset W_2 \subset Y_3], [V_4 \subset W_6 \subset Y_4]\}, \\
\mathcal{F}_6 &= \{[V_1 \subset W_1 \subset Y_2], [V_2 \subset W_4 \subset Y_3], [V_3 \subset W_2 \subset Y_1], [V_4 \subset W_6 \subset Y_4]\}, \\
\mathcal{F}_7 &= \{[V_1 \subset W_1 \subset Y_1], [V_2 \subset W_4 \subset Y_2], [V_3 \subset W_6 \subset Y_3], [V_4 \subset W_3 \subset Y_4]\}, \\
\mathcal{F}_8 &= \{[V_1 \subset W_1 \subset Y_2], [V_2 \subset W_4 \subset Y_3], [V_3 \subset W_6 \subset Y_4], [V_4 \subset W_3 \subset Y_1]\}, \\
\mathcal{F}_9 &= \{[V_1 \subset W_1 \subset Y_1], [V_2 \subset W_4 \subset Y_2], [V_3 \subset W_6 \subset Y_3], [V_4 \subset W_5 \subset Y_4]\}, \\
\mathcal{F}_{10} &= \{[V_1 \subset W_1 \subset Y_1], [V_2 \subset W_4 \subset Y_3], [V_3 \subset W_6 \subset Y_4], [V_4 \subset W_5 \subset Y_2]\}, \\
\mathcal{F}_{11} &= \{[V_1 \subset W_1 \subset Y_1], [V_2 \subset W_5 \subset Y_2], [V_3 \subset W_2 \subset Y_3], [V_4 \subset W_3 \subset Y_4]\}, \\
\mathcal{F}_{12} &= \{[V_1 \subset W_1 \subset Y_2], [V_2 \subset W_5 \subset Y_4], [V_3 \subset W_2 \subset Y_3], [V_4 \subset W_3 \subset Y_1]\}, \\
\mathcal{F}_{13} &= \{[V_1 \subset W_1 \subset Y_1], [V_2 \subset W_5 \subset Y_2], [V_3 \subset W_2 \subset Y_3], [V_4 \subset W_6 \subset Y_4]\}, \\
\mathcal{F}_{14} &= \{[V_1 \subset W_1 \subset Y_2], [V_2 \subset W_5 \subset Y_4], [V_3 \subset W_2 \subset Y_1], [V_4 \subset W_6 \subset Y_3]\}, \\
\mathcal{F}_{15} &= \{[V_1 \subset W_1 \subset Y_1], [V_2 \subset W_5 \subset Y_2], [V_3 \subset W_4 \subset Y_3], [V_4 \subset W_3 \subset Y_4]\}, \\
\mathcal{F}_{16} &= \{[V_1 \subset W_1 \subset Y_2], [V_2 \subset W_5 \subset Y_4], [V_3 \subset W_4 \subset Y_3], [V_4 \subset W_3 \subset Y_1]\}, \\
\mathcal{F}_{17} &= \{[V_1 \subset W_1 \subset Y_1], [V_2 \subset W_5 \subset Y_2], [V_3 \subset W_4 \subset Y_3], [V_4 \subset W_6 \subset Y_4]\}, \\
\mathcal{F}_{18} &= \{[V_1 \subset W_1 \subset Y_1], [V_2 \subset W_5 \subset Y_4], [V_3 \subset W_4 \subset Y_2], [V_4 \subset W_6 \subset Y_3]\}, \\
\mathcal{F}_{19} &= \{[V_1 \subset W_1 \subset Y_1], [V_2 \subset W_5 \subset Y_2], [V_3 \subset W_6 \subset Y_3], [V_4 \subset W_3 \subset Y_4]\}, \\
\mathcal{F}_{20} &= \{[V_1 \subset W_1 \subset Y_2], [V_2 \subset W_5 \subset Y_4], [V_3 \subset W_6 \subset Y_3], [V_4 \subset W_3 \subset Y_1]\}, \\
\mathcal{F}_{21} &= \{[V_1 \subset W_2 \subset Y_1], [V_2 \subset W_1 \subset Y_2], [V_3 \subset W_4 \subset Y_3], [V_4 \subset W_3 \subset Y_4]\}, \\
\mathcal{F}_{22} &= \{[V_1 \subset W_2 \subset Y_3], [V_2 \subset W_1 \subset Y_1], [V_3 \subset W_4 \subset Y_2], [V_4 \subset W_3 \subset Y_4]\}, \\
\mathcal{F}_{23} &= \{[V_1 \subset W_2 \subset Y_1], [V_2 \subset W_1 \subset Y_2], [V_3 \subset W_4 \subset Y_3], [V_4 \subset W_5 \subset Y_4]\}, \\
\mathcal{F}_{24} &= \{[V_1 \subset W_2 \subset Y_3], [V_2 \subset W_1 \subset Y_1], [V_3 \subset W_4 \subset Y_2], [V_4 \subset W_5 \subset Y_4]\}, \\
\mathcal{F}_{25} &= \{[V_1 \subset W_2 \subset Y_1], [V_2 \subset W_1 \subset Y_2], [V_3 \subset W_4 \subset Y_3], [V_4 \subset W_6 \subset Y_4]\}, \\
\mathcal{F}_{26} &= \{[V_1 \subset W_2 \subset Y_3], [V_2 \subset W_1 \subset Y_1], [V_3 \subset W_4 \subset Y_2], [V_4 \subset W_6 \subset Y_4]\}, \\
\mathcal{F}_{27} &= \{[V_1 \subset W_2 \subset Y_1], [V_2 \subset W_1 \subset Y_2], [V_3 \subset W_6 \subset Y_3], [V_4 \subset W_3 \subset Y_4]\}, \\
\mathcal{F}_{28} &= \{[V_1 \subset W_2 \subset Y_3], [V_2 \subset W_1 \subset Y_2], [V_3 \subset W_6 \subset Y_4], [V_4 \subset W_3 \subset Y_1]\}, \\
\mathcal{F}_{29} &= \{[V_1 \subset W_2 \subset Y_1], [V_2 \subset W_1 \subset Y_2], [V_3 \subset W_6 \subset Y_3], [V_4 \subset W_5 \subset Y_4]\}, \\
\mathcal{F}_{30} &= \{[V_1 \subset W_2 \subset Y_3], [V_2 \subset W_1 \subset Y_1], [V_3 \subset W_6 \subset Y_4], [V_4 \subset W_5 \subset Y_2]\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_{31} &= \{[V_1 \subset W_2 \subset Y_1], [V_2 \subset W_4 \subset Y_2], [V_3 \subset W_6 \subset Y_3], [V_4 \subset W_3 \subset Y_4]\},\\
\mathcal{F}_{32} &= \{[V_1 \subset W_2 \subset Y_3], [V_2 \subset W_4 \subset Y_2], [V_3 \subset W_6 \subset Y_4], [V_4 \subset W_3 \subset Y_1]\},\\
\mathcal{F}_{33} &= \{[V_1 \subset W_2 \subset Y_1], [V_2 \subset W_4 \subset Y_2], [V_3 \subset W_6 \subset Y_3], [V_4 \subset W_5 \subset Y_4]\},\\
\mathcal{F}_{34} &= \{[V_1 \subset W_2 \subset Y_1], [V_2 \subset W_4 \subset Y_3], [V_3 \subset W_6 \subset Y_4], [V_4 \subset W_5 \subset Y_2]\},\\
\mathcal{F}_{35} &= \{[V_1 \subset W_2 \subset Y_1], [V_2 \subset W_5 \subset Y_2], [V_3 \subset W_4 \subset Y_3], [V_4 \subset W_3 \subset Y_4]\},\\
\mathcal{F}_{36} &= \{[V_1 \subset W_2 \subset Y_3], [V_2 \subset W_5 \subset Y_4], [V_3 \subset W_4 \subset Y_2], [V_4 \subset W_3 \subset Y_1]\},\\
\mathcal{F}_{37} &= \{[V_1 \subset W_2 \subset Y_1], [V_2 \subset W_5 \subset Y_2], [V_3 \subset W_4 \subset Y_3], [V_4 \subset W_6 \subset Y_4]\},\\
\mathcal{F}_{38} &= \{[V_1 \subset W_2 \subset Y_1], [V_2 \subset W_5 \subset Y_4], [V_3 \subset W_4 \subset Y_2], [V_4 \subset W_6 \subset Y_3]\},\\
\mathcal{F}_{39} &= \{[V_1 \subset W_2 \subset Y_1], [V_2 \subset W_5 \subset Y_2], [V_3 \subset W_6 \subset Y_3], [V_4 \subset W_3 \subset Y_4]\},\\
\mathcal{F}_{40} &= \{[V_1 \subset W_2 \subset Y_3], [V_2 \subset W_5 \subset Y_2], [V_3 \subset W_6 \subset Y_4], [V_4 \subset W_3 \subset Y_1]\},\\
\mathcal{F}_{41} &= \{[V_1 \subset W_3 \subset Y_1], [V_2 \subset W_1 \subset Y_2], [V_3 \subset W_2 \subset Y_3], [V_4 \subset W_5 \subset Y_4]\},\\
\mathcal{F}_{42} &= \{[V_1 \subset W_3 \subset Y_4], [V_2 \subset W_1 \subset Y_1], [V_3 \subset W_2 \subset Y_3], [V_4 \subset W_5 \subset Y_2]\},\\
\mathcal{F}_{43} &= \{[V_1 \subset W_3 \subset Y_1], [V_2 \subset W_1 \subset Y_2], [V_3 \subset W_2 \subset Y_3], [V_4 \subset W_6 \subset Y_4]\},\\
\mathcal{F}_{44} &= \{[V_1 \subset W_3 \subset Y_4], [V_2 \subset W_1 \subset Y_2], [V_3 \subset W_2 \subset Y_1], [V_4 \subset W_6 \subset Y_3]\},\\
\mathcal{F}_{45} &= \{[V_1 \subset W_3 \subset Y_1], [V_2 \subset W_1 \subset Y_2], [V_3 \subset W_4 \subset Y_3], [V_4 \subset W_5 \subset Y_4]\},\\
\mathcal{F}_{46} &= \{[V_1 \subset W_3 \subset Y_4], [V_2 \subset W_1 \subset Y_1], [V_3 \subset W_4 \subset Y_3], [V_4 \subset W_5 \subset Y_2]\},\\
\mathcal{F}_{47} &= \{[V_1 \subset W_3 \subset Y_1], [V_2 \subset W_1 \subset Y_2], [V_3 \subset W_4 \subset Y_3], [V_4 \subset W_6 \subset Y_4]\},\\
\mathcal{F}_{48} &= \{[V_1 \subset W_3 \subset Y_4], [V_2 \subset W_1 \subset Y_1], [V_3 \subset W_4 \subset Y_2], [V_4 \subset W_6 \subset Y_3]\},\\
\mathcal{F}_{49} &= \{[V_1 \subset W_3 \subset Y_1], [V_2 \subset W_1 \subset Y_2], [V_3 \subset W_6 \subset Y_3], [V_4 \subset W_5 \subset Y_4]\},\\
\mathcal{F}_{50} &= \{[V_1 \subset W_3 \subset Y_4], [V_2 \subset W_1 \subset Y_1], [V_3 \subset W_6 \subset Y_3], [V_4 \subset W_5 \subset Y_2]\},\\
\mathcal{F}_{51} &= \{[V_1 \subset W_3 \subset Y_1], [V_2 \subset W_4 \subset Y_2], [V_3 \subset W_2 \subset Y_3], [V_4 \subset W_5 \subset Y_4]\},\\
\mathcal{F}_{52} &= \{[V_1 \subset W_3 \subset Y_4], [V_2 \subset W_4 \subset Y_3], [V_3 \subset W_2 \subset Y_1], [V_4 \subset W_5 \subset Y_2]\},\\
\mathcal{F}_{53} &= \{[V_1 \subset W_3 \subset Y_1], [V_2 \subset W_4 \subset Y_2], [V_3 \subset W_2 \subset Y_3], [V_4 \subset W_6 \subset Y_4]\},\\
\mathcal{F}_{54} &= \{[V_1 \subset W_3 \subset Y_4], [V_2 \subset W_4 \subset Y_2], [V_3 \subset W_2 \subset Y_1], [V_4 \subset W_6 \subset Y_3]\},\\
\mathcal{F}_{55} &= \{[V_1 \subset W_3 \subset Y_1], [V_2 \subset W_4 \subset Y_2], [V_3 \subset W_6 \subset Y_3], [V_4 \subset W_5 \subset Y_4]\},\\
\mathcal{F}_{56} &= \{[V_1 \subset W_3 \subset Y_1], [V_2 \subset W_4 \subset Y_3], [V_3 \subset W_6 \subset Y_4], [V_4 \subset W_5 \subset Y_2]\},\\
\mathcal{F}_{57} &= \{[V_1 \subset W_3 \subset Y_1], [V_2 \subset W_5 \subset Y_2], [V_3 \subset W_2 \subset Y_3], [V_4 \subset W_6 \subset Y_4]\},\\
\mathcal{F}_{58} &= \{[V_1 \subset W_3 \subset Y_4], [V_2 \subset W_5 \subset Y_2], [V_3 \subset W_2 \subset Y_1], [V_4 \subset W_6 \subset Y_3]\},\\
\mathcal{F}_{59} &= \{[V_1 \subset W_3 \subset Y_1], [V_2 \subset W_5 \subset Y_2], [V_3 \subset W_4 \subset Y_3], [V_4 \subset W_6 \subset Y_4]\},\\
\mathcal{F}_{60} &= \{[V_1 \subset W_3 \subset Y_1], [V_2 \subset W_5 \subset Y_4], [V_3 \subset W_4 \subset Y_2], [V_4 \subset W_6 \subset Y_3]\}.
\end{aligned}
$$

Observation: This new approach corresponds to the limit case with $q = 1$ of the flag codes.

Idea: Finding optimal cardinalities and characterizations seems easier for some small parameters.

Question: How many admissible type vectors exist for a set with n elements?

In the set flag codes, as the proper subsets will determine the code, we can have

Table 4.1: Maximal Set Flag Codes

$\|S\|$	type	d_{set_f}	Size of the maximal set flag code	Number of the maximal set flag codes
3	$(1,2)$ *	4	$3 = (q^2+q+1)\|_{q=1}$	2
4	$(1,2)$	4	$4 = (q^3+q^2+q+1)\|_{q=1}$	30
4	$(1,3)$	4	$4 = (q^3+q^2+q+1)\|_{q=1}$	9
4	$(2,3)$	4	$4 = (q^3+q^2+q+1)\|_{q=1}$	30
4	$(1,2,3)$ *	6	$4 = (q^3+q^2+q+1)\|_{q=1}$	60
5	$(1,4)$	4	$5 = (q^4+q^3+q^2+q+1)\|_{q=1}$	44
5	$(2,3)$	4	$10 = (q^6+q^5+2q^4+2q^3+2q^2+q+1)\|_{q=1}$	60
5	$(1,2,3)$	6	$5 = (q^4+q^3+q^2+q+1)\|_{q=1}$	43632
6	$(1,5)$	4	$6 = (q^5+q^4+q^3+q^2+q+1)\|_{q=1}$	265
6	$(2,4)$	4	$14 = (q^8+q^7+2q^6+2q^5+2q^4+2q^3+2q^2+q+1)\|_{q=1}$	3013854
7	$(1,6)$	4	$7 = (q^6+q^5+q^4+q^3+q^2+q+1)\|_{q=1}$	1854
8	$(1,7)$	4	$8 = (q^7+q^6+q^5+q^4+q^3+q^2+q+1)\|_{q=1}$	14833
9	$(1,8)$	4	$9 = (q^8+q^7+q^6+q^5+q^4+q^3+q^2+q+1)\|_{q=1}$	133496

* full flags.

$(2^{n-1}-1)$ type vectors as we exclude the empty set. All the calculations for maximal flag codes can be operated more easily for the set flag codes and this also reduces the need for computational work and the necessary space for the results.

Theorem 4. *If we are given a type vector $T^* := (t_1, \ldots, t_r)$ with the set flag distance $2r$, the set flag code in a set S with n elements exists with cardinality of $\min\{\binom{n}{t_1}, \ldots, \binom{n}{t_r}\}$ and the number of such maximal set flag codes can be bounded as*

$$|\mathcal{M}_{S,n}(T^*; 2r)| \leq \prod_{i}^{N} (\gamma_i!)^{\gamma_i}, \tag{4.1}$$

where γ_i is the sum of marginals of the corresponding tensor and N is the total number of vertices of the Grassmannian Correspondence Hypergraph.

CHAPTER 5

EQUIVALENT CODES

Take π as a permutation of $\{1,\ldots,n\}$ and $\alpha_1,\ldots,\alpha_n$ non-zero elements of $\mathbb{F}_q$ along with $\varphi=\varphi_{\alpha_1,\ldots,\alpha_n}$ as diagonal matrix with entries $\alpha_1,\ldots,\alpha_n$. Therefore, $\psi=\varphi\circ\pi$ is a monomial transformation on $\mathbb{F}_q^n$.

Observation: $[V_1\subseteq V_2\subseteq\cdots\subseteq V_s]$ is a flag of type $(t_1,t_2,\ldots,t_s)$ in $\mathbb{F}_q^n$ if and only if $[\psi(V_1)\subseteq\psi(V_2)\subseteq\cdots\subseteq\psi(V_s)]$ is a flag of type $(t_1,t_2,\ldots,t_s)$ in $\mathbb{F}_q^n$.

Observation: $\mathcal{C}$ is a flag code of type $(t_1,t_2,\ldots,t_s)$ with $d_f(\mathcal{C})=d$ if and only if $\psi(\mathcal{C})$ is a flag code of type $(t_1,t_2,\ldots,t_s)$ with $d_f(\psi(\mathcal{C}))=d$.

We call $\mathcal{C}$ and $\psi(\mathcal{C})$ are monomially equivalent and denote $\mathcal{C}\underset{\psi}{\sim}\psi(\mathcal{C})$.

For (6.7) monomially equivalent of the ones that can be obtained by the work of [23] are listed as:

$$\begin{array}{lll}
\mathcal{F}_1 \underset{\psi_1}{\rightarrow} \mathcal{F}_1, & \mathcal{F}_2 \underset{\psi_1}{\rightarrow} \mathcal{F}_2, & \mathcal{F}_3 \underset{\psi_1}{\rightarrow} \mathcal{F}_3,\\
\mathcal{F}_1 \underset{\psi_2}{\rightarrow} \mathcal{F}_{23}, & \mathcal{F}_2 \underset{\psi_2}{\rightarrow} \mathcal{F}_{12}, & \mathcal{F}_3 \underset{\psi_2}{\rightarrow} \mathcal{F}_4,\\
\mathcal{F}_1 \underset{\psi_3}{\rightarrow} \mathcal{F}_{11}, & \mathcal{F}_2 \underset{\psi_3}{\rightarrow} \mathcal{F}_{24}, & \mathcal{F}_3 \underset{\psi_3}{\rightarrow} \mathcal{F}_6,\\
\mathcal{F}_1 \underset{\psi_4}{\rightarrow} \mathcal{F}_{20}, & \mathcal{F}_2 \underset{\psi_4}{\rightarrow} \mathcal{F}_{15}, & \mathcal{F}_3 \underset{\psi_4}{\rightarrow} \mathcal{F}_5,\\
\mathcal{F}_1 \underset{\psi_5}{\rightarrow} \mathcal{F}_7, & \mathcal{F}_2 \underset{\psi_5}{\rightarrow} \mathcal{F}_{17}, & \mathcal{F}_3 \underset{\psi_5}{\rightarrow} \mathcal{F}_{19},\\
\mathcal{F}_1 \underset{\psi_6}{\rightarrow} \mathcal{F}_{16}, & \mathcal{F}_2 \underset{\psi_6}{\rightarrow} \mathcal{F}_8, & \mathcal{F}_3 \underset{\psi_6}{\rightarrow} \mathcal{F}_{18},
\end{array}$$

where $\psi_1=I,\ \psi_2=(12),\ \psi_3=(13),\ \psi_4=(23),\ \psi_5=(12)(13),\ \psi_6=(12)(23)$.

Result: The orbits of $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ can be listed as follows

$$\begin{aligned}
\overline{\mathcal{F}_1} &= \{\mathcal{F}_1, \mathcal{F}_{23}, \mathcal{F}_{11}, \mathcal{F}_{20}, \mathcal{F}_7, \mathcal{F}_{16}\}, \\
\overline{\mathcal{F}_2} &= \{\mathcal{F}_2, \mathcal{F}_{12}, \mathcal{F}_{24}, \mathcal{F}_{15}, \mathcal{F}_{17}, \mathcal{F}_8\}, \\
\overline{\mathcal{F}_3} &= \{\mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_6, \mathcal{F}_5, \mathcal{F}_{19}, \mathcal{F}_{18}\}.
\end{aligned}$$

$$\overline{\mathcal{F}_{22}} = \{\mathcal{F}_{22}, \mathcal{F}_{10}, \mathcal{F}_{14}, \mathcal{F}_{21}, \mathcal{F}_9, \mathcal{F}_{13}\}.$$

The latter one is a new class derived from our extension. If we investigate a similar case for (6.24), the number of monomially equivalent ones to the results of [23] will be $5.4! = 120$. We found 328672649760 of them in our search. The remaining ones also can be divided into equivalency classes as shown here.

CHAPTER 6

CHARACTERIZATION OF ALL MAXIMAL FLAG CODES OF TYPE-$(1,2)$ IN $\mathbb{F}_q^3$ and $\mathbb{F}_q^4$ WITH $d=4$ FOR SMALL q

Let $\mathbb{F}_q$ be a finite field, let $n = 3$, $T = (1,2)$ and $d = 4$. Using the information provided by [23], we obtain that

$$\mathcal{A}_{\mathbb{F}_q,3}((1,2);4) = q^2+q+1.$$

Let $\mathcal{C}$ be a maximal flag code of type-$(1,2)$ with $d=4$ in $\mathbb{F}_q^3$ (see Definition 9). Note that $\mathcal{C}$ is also full flag code in $\mathbb{F}_q^3$ over $\mathbb{F}_q$ (see Definition 7) and $d=4$ satisfies (2.2) with $n=3$. Note that the number of distinct 2-dimensional $\mathbb{F}_q$-linear subspaces in $\mathbb{F}_q^3$ is

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = q^2+q+1.$$

Let $N = q^2+q+1$ and $W_1,\ldots,W_N$ be a fixed enumeration of 2-dimensional $\mathbb{F}_q$-linear subspaces in $\mathbb{F}_q^3$.

Let $[U_1 \subset U_2], [V_1 \subset V_2] \in \mathcal{C}$ be two distinct flags. We observe that

$$\begin{aligned} d_f([U_1 \subset U_2]\,, \quad [V_1 \subset V_2]) &= d_S(U_1,V_1) + d_S(U_2,V_2) \\ \text{and} \qquad d_S(U_1,V_1) &\le 2, \quad d_S(U_2,V_2) \le 2 \end{aligned} \tag{6.1}$$

where

$$d_S(U_1,V_1) = 2.1 - 2\dim(U_1\cap V_1) \text{ and } d_S(U_2,V_2) = 2.2 - 2\dim(U_2\cap V_2). \tag{6.2}$$

Moreover,

$$\dim(U_1\cap V_1) \ge 1 \;\text{ and }\; (\dim(U_2\cap V_2) = 1 \iff U_2 \ne V_2)\,. \tag{6.3}$$

The last statement follows from the observation

$$\dim(\langle U_2, V_2\rangle) + \dim(U_2 \cap V_2) = 2 + 2 \text{ and } \langle U_2, V_2\rangle \subset \mathbb{F}_q^3.$$

Combining (6.1), (6.2) and (6.3), we obtain that

$$U_1 \neq V_1 \text{ and } U_2 \neq V_2.$$

Recall that we choose and fix the enumeration $W_1, \ldots, W_N$ of 2-dimensional $\mathbb{F}_q$-linear subspaces $\mathbb{F}_q^3$. We further choose and fix an enumeration of 1-dimensional $\mathbb{F}_q$-linear subspaces $\mathbb{F}_q^3$ as $V_1, \ldots, V_N$. These arguments imply that a maximal flag code $\mathcal{C}$ of type-$(1,2)$ with $d = 4$ in $\mathbb{F}_q^3$ is represented uniquely as an N-tuple.

$$\mathcal{C} = [W_{i_1}, W_{i_2}, \ldots, W_{i_N}] \tag{6.4}$$

where

$$V_1 \subset W_{i_1}, V_2 \subset W_{i_2}, \ldots, V_N \subset W_{i_N} \tag{6.5}$$

and

$$(i_1, i_2, \ldots, i_N) \text{ is a permutation of } (1, 2, \ldots, N). \tag{6.6}$$

Hence, the problem of finding a maximal flag code of type-$(1,2)$ with $d = 4$ in $\mathbb{F}_q^3$ is exactly finding a permutation $(i_1, i_2, \ldots, i_N)$ of $(1, 2, \ldots, N)$ as in (6.6) such that (6.5) holds. We solve this problem completely if $q = 2$ and $q = 3$ using an exhaustive computer search via MAGMA [8] in the following two theorems.

First, we consider the case of $q = 2$.

Theorem 5. *Let $\mathbb{F}_q$ be a finite field with $q = 2$. Let $n = 3, T = (1,2)$ and $d = 4$. Let $N = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = q^2 + q + 1 = 7$ be the number of distinct 2-dimensional subspaces in $\mathbb{F}_q^3$. Note that N is also equal to the number of distinct 1-dimensional subspaces of $\mathbb{F}_q^3$. Let $V_1, \ldots, V_N$ be an enumeration of all 1-dimensional subspaces of $\mathbb{F}_q^3$ given explicitly as follows:*

$$V_1 = \langle(0,0,1)\rangle, \quad V_2 = \langle(0,1,0)\rangle, \quad V_3 = \langle(1,0,0)\rangle, \quad V_4 = \langle(0,1,1)\rangle,$$
$$V_5 = \langle(1,1,0)\rangle, \quad V_6 = \langle(1,1,1)\rangle, \quad V_7 = \langle(1,0,1)\rangle.$$

Let $W_1, \ldots, W_N$ be an enumeration of all 2-dimensional subspaces of $\mathbb{F}_q^3$ given explicitly as follows:

$$W_1 = rs\begin{pmatrix}1\ 0\ 0\\ 0\ 1\ 0\end{pmatrix}, W_2 = rs\begin{pmatrix}1\ 0\ 0\\ 0\ 1\ 1\end{pmatrix}, W_3 = rs\begin{pmatrix}1\ 0\ 1\\ 0\ 1\ 0\end{pmatrix}, W_4 = rs\begin{pmatrix}1\ 0\ 1\\ 0\ 1\ 1\end{pmatrix},$$
$$W_5 = rs\begin{pmatrix}1\ 0\ 0\\ 0\ 0\ 1\end{pmatrix}, W_6 = rs\begin{pmatrix}1\ 1\ 0\\ 0\ 0\ 1\end{pmatrix}, W_7 = rs\begin{pmatrix}0\ 1\ 0\\ 0\ 0\ 1\end{pmatrix}.$$

Here, rs denotes the row space of the corresponding 2×3 matrix over $\mathbb{F}_q$.

Under the notation of (6.4), (6.5), (6.6), the set $\mathcal{M}_{\mathbb{F}_2,3}((1,2);4)$ of maximal flag codes of type-$(1,2)$ in $\mathbb{F}_q^3$ is exactly the set of 24 flag codes $\mathcal{F}_1, \ldots, \mathcal{F}_{24}$ given explicitly as follows:

$$\begin{aligned}
\mathcal{F}_1 &= [W_7, W_1, W_2, W_4, W_6, W_3, W_5], \\
\mathcal{F}_2 &= [W_6, W_3, W_5, W_7, W_1, W_2, W_4], \\
\mathcal{F}_3 &= [W_5, W_7, W_1, W_2, W_4, W_6, W_3], \\
\mathcal{F}_4 &= [W_7, W_1, W_5, W_2, W_4, W_6, W_3], \\
\mathcal{F}_5 &= [W_7, W_1, W_5, W_2, W_6, W_3, W_4], \\
\mathcal{F}_6 &= [W_7, W_1, W_5, W_4, W_6, W_2, W_3], \\
\mathcal{F}_7 &= [W_7, W_3, W_5, W_2, W_1, W_6, W_4], \\
\mathcal{F}_8 &= [W_7, W_3, W_2, W_4, W_1, W_6, W_5], \\
\mathcal{F}_9 &= [W_7, W_3, W_1, W_2, W_4, W_6, W_5], \\
\mathcal{F}_{10} &= [W_7, W_3, W_1, W_4, W_6, W_2, W_5], \\
\mathcal{F}_{11} &= [W_6, W_7, W_1, W_2, W_4, W_3, W_5], \\
\mathcal{F}_{12} &= [W_6, W_7, W_2, W_4, W_1, W_3, W_5],
\end{aligned} \tag{6.7}$$

$$
\begin{aligned}
\mathcal{F}_{13} &= [W_6, W_7, W_5, W_4, W_1, W_2, W_3]\,, \\
\mathcal{F}_{14} &= [W_6, W_7, W_5, W_2, W_1, W_3, W_4]\,, \\
\mathcal{F}_{15} &= [W_6, W_3, W_1, W_7, W_4, W_2, W_5]\,, \\
\mathcal{F}_{16} &= [W_6, W_1, W_5, W_7, W_4, W_2, W_3]\,, \\
\mathcal{F}_{17} &= [W_6, W_1, W_2, W_7, W_4, W_3, W_5]\,, \\
\mathcal{F}_{18} &= [W_5, W_7, W_1, W_2, W_6, W_3, W_4]\,, \\
\mathcal{F}_{19} &= [W_5, W_7, W_1, W_4, W_6, W_2, W_3]\,, \\
\mathcal{F}_{20} &= [W_5, W_7, W_2, W_4, W_1, W_6, W_3]\,, \\
\mathcal{F}_{21} &= [W_5, W_1, W_2, W_7, W_4, W_6, W_3]\,, \\
\mathcal{F}_{22} &= [W_5, W_1, W_2, W_7, W_6, W_3, W_4]\,, \\
\mathcal{F}_{23} &= [W_5, W_3, W_1, W_7, W_6, W_2, W_4]\,, \\
\mathcal{F}_{24} &= [W_5, W_3, W_2, W_7, W_1, W_6, W_4]\,.
\end{aligned}
$$

Proof. There are exactly $7! = 5040$ permutations of $(1, \ldots, N) = (1, \ldots, 7)$. For each permutation $\pi = (i_1, \ldots, i_N)$ of $(1, \ldots, 7)$ we check if (6.5) holds. By MAGMA, we obtain that the permutations $\pi = (i_1, \ldots, i_N)$ of $(1, \ldots, 7)$ satisfying (6.5) are exactly the ones corresponding to $\mathcal{F}_1, \ldots, \mathcal{F}_{24}$.

□

Let g be a generator of $\mathbb{F}_{2^3}^*$ and let $\hat{W}_1, \ldots, \hat{W}_7$ be all 2-dimensional subspaces of $\mathbb{F}_{2^3}$. There exists $1 \leq i \leq 7$ such that $1 \in W_i$, by renumbering let $1 \in \hat{W}_1$. Let $\hat{W}(j) = \langle 1, g^j \rangle$ for $1 \leq j \leq 6$. As $g^3 = g + 1$, we observe that

$$
\begin{aligned}
\langle 1, g \rangle &= \langle 1, g^3 \rangle\,, \\
\langle 1, g^2 \rangle &= \langle 1, g^6 \rangle\,, \\
\langle 1, g^4 \rangle &= \langle 1, g^5 \rangle\,.
\end{aligned}
$$

Let $\mathcal{J} = \{1, 2, 4\}$. Note that $\langle 1, g^{j_1} \rangle \neq \langle 1, g^{j_2} \rangle$ and $\dim_{\mathbb{F}_q}(\langle 1, g^{j_1} \rangle) = \dim_{\mathbb{F}_q}(\langle 1, g^{j_2} \rangle) = 2$ if $j_1, j_2 \in \mathcal{J}$ and $j_1 \neq j_2$. For $j \in \mathcal{J}$, let $\mathcal{C}(j)$ be the collection given by

$$
\mathcal{C}(j) = \left\{ \left[\langle 1 \rangle, \hat{W}(j) \right], \left[\langle g \rangle, g\hat{W}(j) \right], \ldots, \left[\langle g^6 \rangle, g^6\hat{W}(j) \right] \right\}. \tag{6.8}
$$

Next, we give an explicit version of a result of [23]

Proposition 1. *Let $\mathbb{F}_q$ be a finite field with $q = 2$. Let $n = 3,\ T = (1, 2)$ and $d = 4$. Let g be a generator of $\mathbb{F}_{q^3}^*$ and let $\mathcal{J} = \{1, 2, 4\}$. For each $j \in \mathcal{J}$, the collection $\mathcal{C}(j)$ given in (6.8) is a maximal flag code of type-T in $\mathbb{F}_2^3$ and hence an element of $\mathcal{M}_{\mathbb{F}_2,3}((1, 2); 4)$. Moreover, $\mathcal{C}(j_1) \neq \mathcal{C}(j_2)$ if $j_1, j_2 \in \mathcal{J}$ and $j_1 \neq j_2$.*

Proof. Recall that $\hat{W}(j) = \langle 1, g^j\rangle = \{0, 1, g^j, g^j + 1\}$,

$$\mathcal{C}(j) = \left\{ \left[\langle 1\rangle, \hat{W}(j)\right], \left[\langle g\rangle, g\hat{W}(j)\right], \ldots, \left[\left\langle g^6\right\rangle, g^6\hat{W}(j)\right] \right\}.$$

As g is a generator of $\mathbb{F}_{2^3}^*$, we have $\langle g^{i_1}\rangle = \{0, g^{i_1}\} \neq \{0, g^{i_2}\} = \langle g^{i_2}\rangle$ for $1 \leq i_1 < i_2 \leq 6$. We observe that
$g^{i_1}\hat{W}(j) = \{0, g^{i_1}, g^{i_1+j}, g^{i_1} + g^{i_1+j}\} \neq \{0, g^{i_2}, g^{i_2+j}, g^{i_2} + g^{i_2+j}\} = g^{i_2}\hat{W}(j)$ for $1 \leq i_1 < i_2 \leq 6$. Indeed, otherwise

$$\hat{W}(j) = g^{i_2-i_1}\hat{W}(j).$$

Put $i = i_2 - i_1$. Note that $1 \leq i \leq 6$. We have

$$\begin{aligned} \hat{W}(j) &= \langle 1, g^j\rangle \text{ and} \\ g^i\hat{W}(j) &= \langle g^i, g^{i+j}\rangle. \end{aligned}$$

If $g^i\hat{W}(j) = \hat{W}(j)$, then $1 \in g^i\hat{W}(j)$ and $g^i \in g^i\hat{W}(j)$. These imply

$$\begin{aligned} 1 &= a.g^i + b.g^{i+j}, \\ g^j &= c.g^i + d.g^{i+j}. \end{aligned}$$

for $a, b, c, d \in \mathbb{F}_2$. Hence, we also have

$$1 = c.g^{i-j} + d.g^i = a.g^i + b.g^{i+j}. \tag{6.9}$$

Put $x = g^j$. Dividing (6.9) by g^i, we obtain

$$cx^{-1} + d = a + bx$$

and hence

$$bx^2 + (a + d)x + c = 0. \tag{6.10}$$

Using (6.10), we get a contradiction as $\mathbb{F}_2(x) = \mathbb{F}_{2^3}$ and the minimal polynomial of x over $\mathbb{F}_2$ has degree 3. Next we observe that

$$\{0, g^i\} = \langle g^i \rangle \subset g^i \hat{W}(j) = \{0, g^i, g^{i+j}, g^i + g^{i+j}\}.$$

These arguments show that $\mathcal{C}(j)$ is a maximal flag code of type-T in $\mathbb{F}_{2^3}$.

Finally, we show that $\mathcal{C}(j_1) \neq \mathcal{C}(j_2)$ if $j_1, j_2 \in \mathcal{J}$ and $j_1 \neq j_2$. Indeed
$[\langle 1 \rangle, \langle 1, g^{j_1} \rangle] \in \mathcal{C}(j_1) \setminus \mathcal{C}(j_2)$ as $[\langle 1 \rangle, \langle 1, g^{j_2} \rangle] \in \mathcal{C}(j_2)$ and
$\langle 1, g^{j_1} \rangle \neq \langle 1, g^{j_2} \rangle$.

□

Corollary 1. *Let $\mathbb{F}_q$ be a finite field with $q = 2$. Let $n = 3$, $T = (1, 2)$ and $d = 4$. Let $g \in \mathbb{F}_{2^3}$ with $g^3 + g + 1 = 0$. Then the maximal flag codes of type-$(1, 2)$ in $\mathbb{F}_{2^3}$ obtained by [23] correspond to the subset $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ of the ones given in (6.7).*

Remark 5. *Proposition 1 is an explicit presentation of a construction of [23] for $q = 2$, $n = 3$, $T = (1, 2)$ and $d = 4$. In particular, we show that the number of maximal flag codes of type-$(1, 2)$ for $q = 2$, $n = 3$, $T = (1, 2)$ and $d = 4$ constructed from [23] is exactly 3. Hence, we detect 24 maximal flag codes of type-$(1, 2)$ for $q = 2$, $n = 3$, $T = (1, 2)$ and $d = 4$ in Theorem 5.*

Next, we consider the case of $q = 3$.

Theorem 6. *Let $\mathbb{F}_q$ be a finite field with $q = 3$. Let $n = 3$, $T = (1, 2)$ and $d = 4$. Let $N = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = q^2 + q + 1 = 13$ be the number of distinct 2-dimensional subspaces in $\mathbb{F}_3^3$. Note that N is also equal to the number of distinct 1-dimensional subspaces of $\mathbb{F}_3^3$. Let $V_1, \ldots, V_N$ be an enumeration of all 1-dimensional subspaces of $\mathbb{F}_3^3$ given* *explicitly as follows:*

$$
\begin{aligned}
&V_1 = \langle(0,0,1)\rangle\,, \quad V_2 = \langle(0,1,0)\rangle\,, \quad V_3 = \langle(1,0,0)\rangle\,, \quad V_4 = \langle(0,1,2)\rangle\,,\\
&V_5 = \langle(1,2,0)\rangle\,, \quad V_6 = \langle(2,1,2)\rangle\,, \quad V_7 = \langle(1,1,1)\rangle\,, \quad V_8 = \langle(1,2,2)\rangle\,,\\
&V_9 = \langle(2,0,2)\rangle\,, \quad V_{10} = \langle(0,1,1)\rangle\,, \quad V_{11} = \langle(1,1,0)\rangle\,, \quad V_{12} = \langle(1,1,2)\rangle\,,\\
&V_{13} = \langle(1,0,2)\rangle\,.
\end{aligned}
$$

Let $W_1, \ldots, W_N$ *be an enumeration of all* 2*-dimensional subspaces of* $\mathbb{F}_3^3$ *given explicitly as follows:*

$$
\begin{aligned}
&W_1 = rs\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}, \; W_2 = rs\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}, \; W_3 = rs\begin{pmatrix}1&0&0\\0&1&2\end{pmatrix}, \; W_4 = rs\begin{pmatrix}1&0&1\\0&1&2\end{pmatrix},\\
&W_5 = rs\begin{pmatrix}1&2&0\\0&0&2\end{pmatrix}, \; W_6 = rs\begin{pmatrix}1&0&1\\0&1&0\end{pmatrix}, \; W_7 = rs\begin{pmatrix}2&0&0\\0&2&2\end{pmatrix}, \; W_8 = rs\begin{pmatrix}1&0&2\\0&2&0\end{pmatrix},\\
&W_9 = rs\begin{pmatrix}2&0&2\\0&1&1\end{pmatrix}, W_{10} = rs\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}, W_{11} = rs\begin{pmatrix}1&1&0\\0&0&2\end{pmatrix}, W_{12} = rs\begin{pmatrix}1&0&2\\0&1&0\end{pmatrix},\\
&W_{13} = rs\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}.
\end{aligned}
$$

Here, rs denotes the row space of the corresponding 2×3 *matrix over* $\mathbb{F}_q$*.*

Under the notation of (6.4), (6.5), (6.6), the set $\mathcal{M}_{\mathbb{F}_3,3}((1,2);4)$ *of maximal flag codes of type-*$(1,2)$ *with* $d = 4$ *in* $\mathbb{F}_3^3$ *is exactly the set of* 3852 *flag codes* $\mathcal{F}_1, \ldots, \mathcal{F}_{3852}$ *have been given explicitly detected by an exhaustive search via MAGMA [8], they have been uploaded to the link: Github*

Some of them we put here in our paper as follows:

$$
\begin{aligned}
\mathcal{F}_1 &= [W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{12}, W_{13}]\,,\\
\mathcal{F}_2 &= [W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{12}, W_{13}, W_1, W_2, W_3, W_4]\,,\\
\mathcal{F}_3 &= [W_{11}, W_{12}, W_{13}, W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}]\,,\\
\mathcal{F}_4 &= [W_{13}, W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{12}]\,,\\
\mathcal{F}_5 &= [W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_{12}, W_9, W_{10}, W_8, W_{11}, W_{13}]\,,\\
\mathcal{F}_6 &= [W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_{12}, W_{13}, W_9, W_8, W_{11}, W_{10}]\,,\\
\mathcal{F}_7 &= [W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_{12}, W_8, W_{10}, W_{11}, W_9, W_{13}]\,,\\
\mathcal{F}_8 &= [W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_{12}, W_8, W_9, W_{10}, W_{11}, W_{13}]\,,\\
\mathcal{F}_9 &= [W_1, W_2, W_3, W_4, W_5, W_6, W_{11}, W_8, W_9, W_7, W_{10}, W_{12}, W_{13}]\,,\\
\mathcal{F}_{10} &= [W_1, W_2, W_3, W_4, W_5, W_6, W_{11}, W_8, W_{13}, W_7, W_{10}, W_9, W_{12}]\,,\\
\mathcal{F}_{11} &= [W_1, W_2, W_3, W_4, W_5, W_6, W_{11}, W_{12}, W_{13}, W_7, W_8, W_9, W_{10}]\,,\\
\mathcal{F}_{12} &= [W_1, W_2, W_3, W_4, W_5, W_6, W_{11}, W_{12}, W_8, W_7, W_9, W_{10}, W_{13}]\,,\\
\mathcal{F}_{13} &= [W_1, W_2, W_3, W_4, W_5, W_6, W_{11}, W_7, W_9, W_{10}, W_8, W_{12}, W_{13}]\,,\\
\mathcal{F}_{14} &= [W_1, W_2, W_3, W_4, W_5, W_6, W_{11}, W_7, W_{13}, W_{10}, W_8, W_9, W_{12}]\,,\\
\mathcal{F}_{15} &= [W_1, W_2, W_3, W_4, W_5, W_6, W_{11}, W_7, W_{13}, W_9, W_8, W_{12}, W_{10}]\,,\\
\mathcal{F}_{16} &= [W_1, W_2, W_3, W_4, W_5, W_6, W_{11}, W_7, W_8, W_9, W_{10}, W_{12}, W_{13}]\,,\\
\mathcal{F}_{17} &= [W_1, W_2, W_3, W_4, W_5, W_{10}, W_7, W_8, W_6, W_9, W_{11}, W_{12}, W_{13}]\,,\\
\mathcal{F}_{18} &= [W_1, W_2, W_3, W_4, W_5, W_{10}, W_7, W_{12}, W_6, W_9, W_8, W_{11}, W_{13}]\,,\\
\mathcal{F}_{19} &= [W_1, W_2, W_3, W_4, W_5, W_{10}, W_{11}, W_{12}, W_6, W_7, W_8, W_9, W_{13}]\,,\\
\mathcal{F}_{20} &= [W_1, W_2, W_3, W_4, W_5, W_{10}, W_{11}, W_7, W_6, W_9, W_8, W_{12}, W_{13}]\,,\\
\mathcal{F}_{21} &= [W_1, W_2, W_3, W_4, W_5, W_{10}, W_6, W_8, W_9, W_7, W_{11}, W_{12}, W_{13}]\,,\\
\mathcal{F}_{22} &= [W_1, W_2, W_3, W_4, W_5, W_{10}, W_6, W_8, W_{13}, W_7, W_{11}, W_9, W_{12}]\,,\\
\mathcal{F}_{23} &= [W_1, W_2, W_3, W_4, W_5, W_{10}, W_6, W_{12}, W_9, W_7, W_8, W_{11}, W_{13}]\,,\\
\mathcal{F}_{24} &= [W_1, W_2, W_3, W_4, W_5, W_{10}, W_6, W_{12}, W_8, W_7, W_{11}, W_9, W_{13}]\,,\\
\mathcal{F}_{25} &= [W_1, W_2, W_3, W_4, W_5, W_{10}, W_6, W_7, W_{13}, W_9, W_8, W_{11}, W_{12}]\,,\\
\mathcal{F}_{26} &= [W_1, W_2, W_3, W_4, W_5, W_{10}, W_6, W_7, W_8, W_9, W_{11}, W_{12}, W_{13}]\,,\\
\mathcal{F}_{27} &= [W_1, W_2, W_3, W_4, W_9, W_6, W_7, W_5, W_{13}, W_{10}, W_8, W_{11}, W_{12}]\,,\\
\mathcal{F}_{28} &= [W_1, W_2, W_3, W_4, W_9, W_6, W_7, W_5, W_8, W_{10}, W_{11}, W_{12}, W_{13}]\,,\\
\mathcal{F}_{29} &= [W_1, W_2, W_3, W_4, W_9, W_6, W_{11}, W_5, W_{13}, W_7, W_8, W_{12}, W_{10}]\,,\\
\mathcal{F}_{30} &= [W_1, W_2, W_3, W_4, W_9, W_6, W_{11}, W_5, W_8, W_7, W_{10}, W_{12}, W_{13}]\,.
\end{aligned}
\tag{6.11}
$$

Proof. There are exactly $13!=6227020800$ permutations of $(1,\ldots,N)=(1,\ldots,13)$.

For each permutation $\pi = (i_1, \ldots, i_N)$ of $(1, \ldots, 13)$ we check if (6.5) holds. By MAGMA, we obtain that the permutations $\pi = (i_1, \ldots, i_N)$ of $(1, \ldots, 13)$ satisfying (6.5) are exactly the ones corresponding to $\mathcal{F}_1, \ldots, \mathcal{F}_{3852}$.

□

Let g be a generator of $\mathbb{F}_{3^3}^*$ with $g^3 + 2g + 1 = 0$. Let $\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_{13}$ be all 2-dimensional $\mathbb{F}_3$-linear subspaces of $\mathbb{F}_{3^3}$. There exists $1 \le j \le 13$ such that $1 \in \hat{W}_j$. By renumbering, we assume that $1 \in \hat{W}_1$. Let

$$\hat{W}(j) = \left\langle 1, g^j \right\rangle$$

for $1 \le j \le 12$. Note that $g^{13} \in \mathbb{F}_3^*$. Using $g^3 + 2g + 1 = 0$, we obtain that

$$\begin{aligned}
\langle 1, g \rangle &= \langle 1, g^3 \rangle = \langle 1, g^9 \rangle , \\
\langle 1, g^2 \rangle &= \langle 1, g^8 \rangle = \langle 1, g^{12} \rangle , \\
\langle 1, g^4 \rangle &= \langle 1, g^5 \rangle = \langle 1, g^7 \rangle , \\
\langle 1, g^6 \rangle &= \langle 1, g^{10} \rangle = \langle 1, g^{11} \rangle .
\end{aligned}$$

Let $\mathcal{J} = \{1, 2, 4, 6\}$. Note that for $j \in \mathcal{J}$, let $\mathcal{C}(j)$ be the collection given by

$$\mathcal{C}(j) = \left\{ \left[\langle 1 \rangle , \hat{W}(j) \right], \left[\langle g \rangle , g\hat{W}(j) \right], \ldots, \left[\left\langle g^{12} \right\rangle , g^{12}\hat{W}(j) \right] \right\}. \tag{6.12}$$

Proposition 2. *Let $\mathbb{F}_q$ be a finite field with $q = 3$. Let $n = 3$, $T = (1, 2)$ and $d = 4$. Let g be a generator of $\mathbb{F}_{3^3}^*$ and let $\mathcal{J} = \{1, 2, 4, 6\}$. For each $j \in \mathcal{J}$, the collection $\mathcal{C}(j)$ given in (6.12) is a maximal flag code of type-T in $\mathbb{F}_3^3$ and hence an element of $\mathcal{M}_{\mathbb{F}_3,3}((1, 2); 4)$. Moreover, $\mathcal{C}(j_1) \neq \mathcal{C}(j_2)$ if $j_1, j_2 \in \mathcal{J}$ and $j_1 \neq j_2$.*

Proof. Recall that $\hat{W} = \langle 1, g^j \rangle = \{0, 1, 2, g^j, 2g^j, g^j + 1, g^j + 2, 2g^j + 1, 2g^j + 2\}$ and

$$\mathcal{C}(j) = \left\{ \left[\langle 1 \rangle , \hat{W}(j) \right], \left[\langle g \rangle , g\hat{W}(j) \right], \ldots, \left[\left\langle g^{12} \right\rangle , g^{12}\hat{W}(j) \right] \right\}.$$

As g is a generator of $\mathbb{F}_{3^3}^*$, we have $\langle g^{i_1} \rangle = \{0, g^{i_1}\} \neq \{0, g^{i_2}\} = \langle g^{i_2} \rangle$ for $1 \le i_1 < i_2 \le 12$. We observe that $g^{i_1}\hat{W}(j) \neq \{0, g^{i_2}, g^{i_2+j}, g^{i_2} + g^{i_2+j}\} = g^{i_2}\hat{W}(j)$ for $1 \le i_1 < i_2 \le 12$. Indeed, otherwise

$$\hat{W}(j) = g^{i_2 - i_1}\hat{W}(j).$$

Put $i = i_2 - i_1$. Note that $1 \leq i \leq 12$. We have

$$\begin{aligned} \hat{W}(j) &= \langle 1, g^j \rangle \text{ and} \\ g^i\hat{W}(j) &= \langle g^i, g^{i+j} \rangle . \end{aligned}$$

If $g^i\hat{W}(j) = \hat{W}(j)$, then $1 \in g^i\hat{W}(j)$ and $g^i \in g^i\hat{W}(j)$. These imply

$$\begin{aligned} 1 &= a.g^i + b.g^{i+j}, \\ g^j &= c.g^i + d.g^{i+j}. \end{aligned}$$

for $a, b, c, d \in \mathbb{F}_3$. Hence, we also have

$$1 = c.g^{i-j} + d.g^i = a.g^i + b.g^{i+j}. \tag{6.13}$$

Put $x = g^j$. Dividing (6.13) by g^i, we obtain

$$cx^{-1} + d = a + bx$$

and hence

$$bx^2 + (a + d)x + c = 0. \tag{6.14}$$

Using (6.14), we get a contradiction as $\mathbb{F}_3(x) = \mathbb{F}_{3^3}$ and the minimal polynomial of x over $\mathbb{F}_3$ has degree 3.

Next we observe that

$$\begin{aligned} \{0, g^i\} = \left\langle g^i \right\rangle \subset g^i\hat{W}(j) \\ = \{0, g^i, 2g^i, g^{i+j}, 2g^{i+j}, g^i + g^{i+j}, 2g^i + g^{i+j}, g^i + 2g^{i+j}, 2g^i + 2g^{i+j}\}. \end{aligned}$$

These arguments show that $\mathcal{C}(j)$ is a maximal flag code of type-T in $\mathbb{F}_{3^3}$.

Finally, we show that $\mathcal{C}(j_1) \neq \mathcal{C}(j_2)$ if $j_1, j_2 \in \mathcal{J}$ and $j_1 \neq j_2$. Indeed $[\langle 1 \rangle, \langle 1, g^{j_1} \rangle] \in \mathcal{C}(j_1) \setminus \mathcal{C}(j_2)$ as $[\langle 1 \rangle, \langle 1, g^{j_2} \rangle] \in \mathcal{C}(j_2)$ and $\langle 1, g^{j_1} \rangle \neq \langle 1, g^{j_2} \rangle$.

□

Corollary 2. *Let $\mathbb{F}_q$ be a finite field with $q = 3$. Let $n = 3$, $T = (1, 2)$ and $d = 4$. Let $g \in \mathbb{F}_{3^3}$ with $g^3 + 2g + 1 = 0$. Then the maximal flag codes of type-$(1, 2)$ in $\mathbb{F}_{3^3}$ obtained by [23] correspond to the subset $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4\}$ of the ones given in (6.11).*

Remark 6. *Combining Theorem 6 and Corollary 2, we observe that we detect 3852 maximal flag codes of type-$(1, 2)$, and $d = 4$ for $q = 3$ and $n = 3$ in Theorem 6.*

Next, we consider the case of $q = 3$.

Theorem 7. *Let $\mathbb{F}_q$ be a finite field with $q = 5$. Let $n = 3$, $T = (1, 2)$ and $d = 4$. Let $N = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = q^2 + q + 1 = 31$ be the number of distinct 2-dimensional subspaces in $\mathbb{F}_5^3$. Note that N is also equal to the number of distinct 1-dimensional subspaces of $\mathbb{F}_5^3$. Let $V_1, \ldots, V_N$ be an enumeration of all 1-dimensional subspaces of $\mathbb{F}_5^3$ given explicitly as follows:*

$$\begin{aligned}
&V_1 = \langle(0,0,1)\rangle\,, \quad V_2 = \langle(0,1,0)\rangle\,, \quad V_3 = \langle(0,1,1)\rangle\,, \quad V_4 = \langle(0,1,2)\rangle\,,\\
&V_5 = \langle(0,1,3)\rangle\,, \quad V_6 = \langle(0,1,4)\rangle\,, \quad V_7 = \langle(1,0,0)\rangle\,, \quad V_8 = \langle(1,0,1)\rangle\,,\\
&V_9 = \langle(1,0,2)\rangle\,, \quad V_{10} = \langle(1,0,3)\rangle\,, \quad V_{11} = \langle(1,0,4)\rangle\,, \quad V_{12} = \langle(1,1,0)\rangle\,,\\
&V_{13} = \langle(1,1,1)\rangle\,, \quad V_{14} = \langle(1,1,2)\rangle\,, \quad V_{15} = \langle(1,1,3)\rangle\,, \quad V_{16} = \langle(1,1,4)\rangle\,,\\
&V_{17} = \langle(1,2,0)\rangle\,, \quad V_{18} = \langle(1,2,1)\rangle\,, \quad V_{19} = \langle(1,2,2)\rangle\,, \quad V_{20} = \langle(1,2,3)\rangle\,,\\
&V_{21} = \langle(1,2,4)\rangle\,, \quad V_{22} = \langle(1,3,0)\rangle\,, \quad V_{23} = \langle(1,3,1)\rangle\,, \quad V_{24} = \langle(1,3,2)\rangle\,,\\
&V_{25} = \langle(1,3,3)\rangle\,, \quad V_{26} = \langle(1,3,4)\rangle\,, \quad V_{27} = \langle(1,4,0)\rangle\,, \quad V_{28} = \langle(1,4,1)\rangle\,,\\
&V_{29} = \langle(1,4,2)\rangle\,, \quad V_{30} = \langle(1,4,3)\rangle\,, \quad V_{31} = \langle(1,4,4)\rangle\,.
\end{aligned}$$

Let $W_1, \ldots, W_N$ be an enumeration of all 2-dimensional subspaces of $\mathbb{F}_5^3$ given *explicitly as follows:*

$$W_1 = rs\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix},\ W_2 = rs\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix},\ W_3 = rs\begin{pmatrix}1&1&0\\0&0&1\end{pmatrix},\ W_4 = rs\begin{pmatrix}1&2&0\\0&0&1\end{pmatrix},$$
$$W_5 = rs\begin{pmatrix}1&3&0\\0&0&1\end{pmatrix},\ W_6 = rs\begin{pmatrix}1&4&0\\0&0&1\end{pmatrix},\ W_7 = rs\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix},\ W_8 = rs\begin{pmatrix}1&0&0\\0&1&1\end{pmatrix},$$
$$W_9 = rs\begin{pmatrix}1&0&0\\0&1&2\end{pmatrix},W_{10} = rs\begin{pmatrix}1&0&0\\0&1&3\end{pmatrix},W_{11} = rs\begin{pmatrix}1&0&0\\0&1&4\end{pmatrix},W_{12} = rs\begin{pmatrix}1&0&1\\0&1&0\end{pmatrix},$$
$$W_{13} = rs\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix},W_{14} = rs\begin{pmatrix}1&0&1\\0&1&2\end{pmatrix},W_{15} = rs\begin{pmatrix}1&0&1\\0&1&3\end{pmatrix},W_{16} = rs\begin{pmatrix}1&0&1\\0&1&4\end{pmatrix},$$
$$W_{17} = rs\begin{pmatrix}1&0&2\\0&1&0\end{pmatrix},W_{18} = rs\begin{pmatrix}1&0&2\\0&1&1\end{pmatrix},W_{19} = rs\begin{pmatrix}1&0&2\\0&1&2\end{pmatrix},W_{20} = rs\begin{pmatrix}1&0&2\\0&1&3\end{pmatrix},$$
$$W_{21} = rs\begin{pmatrix}1&0&2\\0&1&4\end{pmatrix},W_{22} = rs\begin{pmatrix}1&0&3\\0&1&0\end{pmatrix},W_{23} = rs\begin{pmatrix}1&0&3\\0&1&1\end{pmatrix},W_{24} = rs\begin{pmatrix}1&0&3\\0&1&2\end{pmatrix},$$
$$W_{25} = rs\begin{pmatrix}1&0&3\\0&1&3\end{pmatrix},W_{26} = rs\begin{pmatrix}1&0&3\\0&1&4\end{pmatrix},W_{27} = rs\begin{pmatrix}1&0&4\\0&1&0\end{pmatrix},W_{28} = rs\begin{pmatrix}1&0&4\\0&1&1\end{pmatrix},$$
$$W_{29} = rs\begin{pmatrix}1&0&4\\0&1&2\end{pmatrix},W_{30} = rs\begin{pmatrix}1&0&4\\0&1&3\end{pmatrix},W_{31} = rs\begin{pmatrix}1&0&4\\0&1&4\end{pmatrix}$$

.

Here, rs denotes the row space of the corresponding 2×3 *matrix over* $\mathbb{F}_q$.

Under the notation of (6.4), (6.5), (6.6), the set $\mathcal{M}_{\mathbb{F}_5,3}((1,2);4)$ *of maximal flag codes of type-*$(1,2)$ *with* $d = 4$ *in* $\mathbb{F}_5^3$ *is exactly the set of* 4598378639550 *flag codes* $\mathcal{F}_1, \ldots, \mathcal{F}_{4598378639550}$ *have been given explicitly detected by an exhaustive search via MAGMA [8]. This much data is not easy to store. Therefore, for a specific purpose, one can get as many maximal flag codes of type-*$(1,2)$ *with* $d = 4$ *in* $\mathbb{F}_5^3$ *from the output of this construction.*

Proof. There are exactly $31! = 8222838654177922817725562880000000$ permutations of $(1, \ldots, N)$=$(1, \ldots, 13)$. For each permutation $\pi = (i_1, \ldots, i_N)$ of $(1, \ldots, 31)$ we check if (6.5) holds. By MAGMA, we obtain that the permutations $\pi = (i_1, \ldots, i_N)$ of $(1, \ldots, 31)$ satisfying (6.5) are exactly the ones corresponding to $\mathcal{F}_1, \ldots, \mathcal{F}_{4598378639550}$.

□

Let g be a generator of $\mathbb{F}_{5^3}^*$ with $g^3 + 3g + 2 = 0$. Let $\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_{31}$ be all 2-dimensional $\mathbb{F}_5$-linear subspaces of $\mathbb{F}_{5^3}$. There exists $1 \leq j \leq 31$ such that $1 \in \hat{W}_j$. By renumbering, we assume that $1 \in \hat{W}_1$. Let

$$\hat{W}(j) = \langle 1, g^j \rangle$$

for $1 \leq j \leq 30$. Note that $g^{31} \in \mathbb{F}_5^*$. Using $g^3 + 3g + 2 = 0$, we obtain that

$$\begin{aligned}
\langle 1, g \rangle &= \langle 1, g^3 \rangle &= \langle 1, g^{10} \rangle &= \langle 1, g^{14} \rangle &= \langle 1, g^{26} \rangle, \\
\langle 1, g^2 \rangle &= \langle 1, g^9 \rangle &= \langle 1, g^{13} \rangle &= \langle 1, g^{25} \rangle &= \langle 1, g^{30} \rangle, \\
\langle 1, g^4 \rangle &= \langle 1, g^{16} \rangle &= \langle 1, g^{21} \rangle &= \langle 1, g^{22} \rangle &= \langle 1, g^{24} \rangle, \\
\langle 1, g^5 \rangle &= \langle 1, g^6 \rangle &= \langle 1, g^8 \rangle &= \langle 1, g^{15} \rangle &= \langle 1, g^{19} \rangle, \\
\langle 1, g^7 \rangle &= \langle 1, g^{11} \rangle &= \langle 1, g^{23} \rangle &= \langle 1, g^{28} \rangle &= \langle 1, g^{29} \rangle, \\
\langle 1, g^{12} \rangle &= \langle 1, g^{17} \rangle &= \langle 1, g^{18} \rangle &= \langle 1, g^{20} \rangle &= \langle 1, g^{27} \rangle,
\end{aligned}$$

.

Let $\mathcal{J} = \{1, 2, 4, 5, 7, 12\}$. Note that for $j \in \mathcal{J}$, let $\mathcal{C}(j)$ be the collection given by

$$\mathcal{C}(j) = \left\{ \left[\langle 1 \rangle, \hat{W}(j) \right], \left[\langle g \rangle, g\hat{W}(j) \right], \ldots, \left[\langle g^{30} \rangle, g^{30}\hat{W}(j) \right] \right\}. \tag{6.15}$$

Proposition 3. *Let $\mathbb{F}_q$ be a finite field with $5 = 3$. Let $n = 3$, $T = (1, 2)$ and $d = 4$. Let g be a generator of $\mathbb{F}_{5^3}^*$ and let $\mathcal{J} = \{1, 2, 4, 5, 7, 12\}$. For each $j \in \mathcal{J}$, the collection $\mathcal{C}(j)$ given in (6.15) is a maximal flag code of type-T in $\mathbb{F}_5^3$ and hence an element of $\mathcal{M}_{\mathbb{F}_{5,3}}((1, 2); 4)$. Moreover, $\mathcal{C}(j_1) \neq \mathcal{C}(j_2)$ if $j_1, j_2 \in \mathcal{J}$ and $j_1 \neq j_2$.*

Proof. Recall that $\hat{W} = \langle 1, g^j \rangle = \{0, 1, 2, 3, 4, g^j, g^j{+}1, g^j{+}2, g^j{+}3, g^j{+}4, 2g^j, 2g^j + 1, 2g^j + 2, 2g^j + 3, 2g^j + 4, 3g^j, 3g^j + 1, 3g^j + 2, 3g^j + 3, 3g^j + 4, 4g^j, 4g^j + 1, 4g^j + 2, 4g^j + 3, 4g^j + 4, \}$ and

$$\mathcal{C}(j) = \left\{ \left[\langle 1 \rangle, \hat{W}(j) \right], \left[\langle g \rangle, g\hat{W}(j) \right], \ldots, \left[\langle g^{30} \rangle, g^{30}\hat{W}(j) \right] \right\}.$$

As g is a generator of $\mathbb{F}_{5^3}^*$, we have $\langle g^{i_1} \rangle = \{0, g^{i_1}\} \neq \{0, g^{i_2}\} = \langle g^{i_2} \rangle$ for $1 \leq i_1 < i_2 \leq 30$. We observe that $g^{i_1}\hat{W}(j) \neq \{0, g^{i_2}, g^{i_2+j}, g^{i_2} + g^{i_2+j}\} = g^{i_2}\hat{W}(j)$ for $1 \leq i_1 < i_2 \leq 30$. Indeed, otherwise

$$\hat{W}(j) = g^{i_2 - i_1}\hat{W}(j).$$

Put $i = i_2 - i_1$. Note that $1 \leq i \leq 30$. We have

$$\begin{aligned}
\hat{W}(j) &= \langle 1, g^j \rangle \text{ and} \\
g^i\hat{W}(j) &= \langle g^i, g^{i+j} \rangle.
\end{aligned}$$

If $g^i\hat{W}(j) = \hat{W}(j)$, then $1 \in g^i\hat{W}(j)$ and $g^i \in g^i\hat{W}(j)$. These imply

$$\begin{aligned} 1 &= a.g^i + b.g^{i+j}, \\ g^j &= c.g^i + d.g^{i+j}. \end{aligned}$$

for $a, b, c, d \in \mathbb{F}_5$. Hence, we also have

$$1 = c.g^{i-j} + d.g^i = a.g^i + b.g^{i+j}. \tag{6.16}$$

Put $x = g^j$. Dividing (6.16) by g^i, we obtain

$$cx^{-1} + d = a + bx$$

and hence

$$bx^2 + (a + d)x + c = 0. \tag{6.17}$$

Using (6.17), we get a contradiction as $\mathbb{F}_5(x) = \mathbb{F}_{5^3}$ and the minimal polynomial of x over $\mathbb{F}_5$ has degree 3.

Next, we observe that

$$\begin{aligned} \{0, g^i\} = \langle g^i \rangle \subset g^i\hat{W}(j) \\ =\{&0, g^i, 2g^i, 3g^i, 4g^i, g^{i+j}, 2g^{i+j}, 3g^{i+j}, 4g^{i+j}, g^i + g^{i+j}, 2g^i + g^{i+j}, \\ &3g^i + g^{i+j}, 4g^i + g^{i+j}, g^i + 2g^{i+j}, 2g^i + 2g^{i+j}, 3g^i + 2g^{i+j}, \\ &4g^i + 2g^{i+j}, g^i + 3g^{i+j}, 2g^i + 3g^{i+j}, 3g^i + 3g^{i+j}, 4g^i + 3g^{i+j}, \\ &g^i + 4g^{i+j}, 2g^i + 4g^{i+j}, 3g^i + 4g^{i+j}, 4g^i + 4g^{i+j}\}. \end{aligned}$$

These arguments show that $\mathcal{C}(j)$ is a maximal flag code of type-T in $\mathbb{F}_{5^3}$.

Finally, we show that $\mathcal{C}(j_1) \neq \mathcal{C}(j_2)$ if $j_1, j_2 \in \mathcal{J}$ and $j_1 \neq j_2$. Indeed $[\langle 1 \rangle, \langle 1, g^{j_1} \rangle] \in \mathcal{C}(j_1) \setminus \mathcal{C}(j_2)$ as $[\langle 1 \rangle, \langle 1, g^{j_2} \rangle] \in \mathcal{C}(j_2)$ and $\langle 1, g^{j_1} \rangle \neq \langle 1, g^{j_2} \rangle$.

□

Corollary 3. *Let $\mathbb{F}_q$ be a finite field with $q = 5$. Let $n = 3$, $T = (1, 2)$ and $d = 4$. Let $g \in \mathbb{F}_{5^3}$ with $g^3 + 2g + 1 = 0$. Then the maximal flag codes of type-$(1, 2)$ in $\mathbb{F}_{5^3}$ obtained by [23] correspond to the subset $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6\}$ of the ones given in **?.*

Remark 7. *Combining Theorem 7 and Corollary 3, we observe that we detect* 4598378639550 *maximal flag codes of type-*$(1,2)$*, and* $d = 4$ *for* $q = 5$ *and* $n = 3$ *in Theorem 7.*

Remark 8. *The main point of the work by Alonso et al. in [4] is to construct flag codes via spreads and the number of collections of certain types of flag codes with a given minimum distance* d *under a fixed vector space* $\mathcal{F}_q^n$ *can be calculated with the help of matcihngs and perfect matchings of graph theory.*

The results we share in this section coincide with the sequence from the online integer encyclopedia [29].

$$\mathbf{24}, \mathbf{3852}, 18534400, \mathbf{4598378639550}.$$

These integers are the numbers of permanents of a projective plane of order n for $q = 2, q = 3, q = 4, q = 5$, respectively. It is also the number of perfect matchings between points and lines in a projective plane. It should be noted that this matching relies on inclusion so that any point is eligible to match with a line such that the chosen point is involved by that line.

6.1 Characterization of All Maximal Flag Codes of type-$(1,2)$ in $\mathbb{F}_q^4$ with $d = 4$ for $q = 2$

Let $\mathbb{F}_q$ be a finite field, let $n = 4$, $T = (1,2)$ and $d = 4$. Using the information provided by [23], we obtain that

$$\mathcal{A}_{\mathbb{F}_q,4}((1,2);4) = q^3 + q^2 + q + 1.$$

Let $\mathcal{C}$ be a maximal flag code of type-$(1,2)$ with $d = 4$ in $\mathbb{F}_q^4$ (see Definition 9). There is an important difference from the case of Section 3. Note that $\mathcal{C}$ is not a full flag. Note that the number N_1 of distinct 1-dimensional $\mathbb{F}_q$-linear subspaces and the number N_2 of distinct 2-dimensional $\mathbb{F}_q$-linear subspaces in $\mathbb{F}_q^4$ are

$$N_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q = q^3 + q^2 + q + 1 \text{ and } N_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = (q^2 + 1)(q^2 + q + 1).$$

Let $[U_1 \subset U_2], [V_1 \subset V_2] \in \mathcal{C}$ be two distinct flags. Note that

$$d_S(U_1, V_1) + d_S(U_2, V_2) \geq 4. \tag{6.18}$$

Assume that $U_1 = V_1$. Then $(U_2 \cap V_2) \supset U_1$ and hence

$$d_S(U_1, V_1) = 0 \text{ and } d_S(U_2, V_2) \leq 2. \tag{6.19}$$

Combining (6.18) and (6.19), we conclude that $U_1 \neq V_1$. If $U_1 \neq V_1$ and $U_2 = V_2$, then

$$d_S(U_1, V_1) + d_S(U_2, V_2) = 2 + 0 = 2. \tag{6.20}$$

Let $V_1, \ldots, V_{N_1}$ be a fixed enumeration of 1-dimensional distinct subspaces of $\mathbb{F}_q^4$. Let $W_1, \ldots, W_{N_2}$ be a fixed enumeration of 2-dimensional distinct subspaces of $\mathbb{F}_q^4$. Using the arguments above, in particular (6.18), (6.19) and (6.20), we obtain that a maximal flag code $\mathcal{C}$ of type-$(1, 2)$ with $d = 4$ in $\mathbb{F}_q^4$ is represented uniquely as an N_1-tuple.

$$\mathcal{C} = \left[W_{i_1}, W_{i_2}, \ldots, W_{i_{N_1}}\right] \tag{6.21}$$

where

$$V_1 \subset W_{i_1}, V_2 \subset W_{i_2}, \ldots, V_{N_1} \subset W_{i_{N_1}} \tag{6.22}$$

and

$$\{i_1, i_2, \ldots, i_{N_1}\} \subset \{1, 2, \ldots, N_2\} \text{ and } i_1, i_2, \ldots, i_{N_1} \text{ are mutually distinct.} \tag{6.23}$$

Hence, the problem of finding a maximal flag code of type-$(1, 2)$ with $d = 4$ in $\mathbb{F}_q^4$ is exactly finding a subset $\mathcal{I} = \{i_1, i_2, \ldots, i_{N_1}\}$ of size N_1 in $\{1, 2, \ldots, N_2\}$ as in (6.20) such that (6.19) holds. We solve this problem completely if $q = 2$ and using exhaustive computer search via MAGMA [8] in the following theorem.

Theorem 8. *Let $\mathbb{F}_q$ be a finite field with $q = 2$. Let $n = 4$, $T = (1, 2)$ and $d = 4$. Let* $N_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q = q^3 + q^2 + q + 1 = 15$ *be the number of distinct 1-dimensional* *subspaces in* $\mathbb{F}_2^4$*. Let* $V_1, \ldots, V_{N_1}$ *be an enumeration of all* 1*-dimensional subspaces of* $\mathbb{F}_2^4$ *given explicitly as follows:*

$$\begin{aligned}
&V_1 = \langle(0,0,0,1)\rangle\,, \quad V_2 = \langle(0,0,1,0)\rangle\,, \quad V_3 = \langle(0,1,0,0)\rangle\,,\\
&V_4 = \langle(1,0,0,0)\rangle\,, \quad V_5 = \langle(0,0,1,1)\rangle\,, \quad V_6 = \langle(0,1,1,0)\rangle\,,\\
&V_7 = \langle(1,1,0,0)\rangle\,, \quad V_8 = \langle(1,0,1,1)\rangle\,, \quad V_9 = \langle(0,1,0,1)\rangle\,,\\
&V_{10} = \langle(1,0,1,0)\rangle\,, \quad V_{11} = \langle(0,1,1,1)\rangle\,, \quad V_{12} = \langle(1,1,1,0)\rangle\,,\\
&V_{13} = \langle(1,1,1,1)\rangle\,, \quad V_{14} = \langle(1,1,0,1)\rangle\,, \quad V_{15} = \langle(1,0,0,1)\rangle\,.
\end{aligned}$$

Note that $N_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = (q^2+1)(q^2+q+1) = 35$ *is the number of distinct* 2*-dimensional subspaces in* $\mathbb{F}_2^4$*. Let* $W_1, \ldots, W_{N_2}$ *be an enumeration of all*

2-dimensional subspaces of $\mathbb{F}_2^4$ given explicitly as follows:

$$W_1 = rs\begin{pmatrix}1\,0\,0\,0\\0\,1\,0\,0\end{pmatrix},\ W_2 = rs\begin{pmatrix}1\,0\,0\,0\\0\,1\,0\,1\end{pmatrix},\ W_3 = rs\begin{pmatrix}1\,0\,0\,0\\0\,1\,1\,0\end{pmatrix},$$

$$W_4 = rs\begin{pmatrix}1\,0\,0\,0\\0\,1\,1\,1\end{pmatrix},\ W_5 = rs\begin{pmatrix}1\,0\,0\,1\\0\,1\,0\,0\end{pmatrix},\ W_6 = rs\begin{pmatrix}1\,0\,0\,1\\0\,1\,0\,1\end{pmatrix},$$

$$W_7 = rs\begin{pmatrix}1\,0\,0\,1\\0\,1\,1\,0\end{pmatrix},\ W_8 = rs\begin{pmatrix}1\,0\,0\,1\\0\,1\,1\,1\end{pmatrix},\ W_9 = rs\begin{pmatrix}1\,0\,1\,0\\0\,1\,0\,0\end{pmatrix},$$

$$W_{10} = rs\begin{pmatrix}1\,0\,1\,0\\0\,1\,0\,1\end{pmatrix}, W_{11} = rs\begin{pmatrix}1\,0\,1\,0\\0\,1\,1\,0\end{pmatrix}, W_{12} = rs\begin{pmatrix}1\,0\,1\,0\\0\,1\,1\,1\end{pmatrix},$$

$$W_{13} = rs\begin{pmatrix}1\,0\,1\,1\\0\,1\,0\,0\end{pmatrix}, W_{14} = rs\begin{pmatrix}1\,0\,1\,1\\0\,1\,0\,1\end{pmatrix}, W_{15} = rs\begin{pmatrix}1\,0\,1\,1\\0\,1\,1\,0\end{pmatrix},$$

$$W_{16} = rs\begin{pmatrix}1\,0\,1\,1\\0\,1\,1\,1\end{pmatrix}, W_{17} = rs\begin{pmatrix}1\,0\,0\,0\\0\,0\,1\,0\end{pmatrix}, W_{18} = rs\begin{pmatrix}1\,0\,0\,1\\0\,0\,1\,0\end{pmatrix},$$

$$W_{19} = rs\begin{pmatrix}1\,1\,0\,0\\0\,0\,1\,0\end{pmatrix}, W_{20} = rs\begin{pmatrix}1\,1\,0\,1\\0\,0\,1\,0\end{pmatrix}, W_{21} = rs\begin{pmatrix}1\,0\,0\,0\\0\,0\,1\,1\end{pmatrix},$$

$$W_{22} = rs\begin{pmatrix}1\,0\,0\,1\\0\,0\,1\,1\end{pmatrix}, W_{23} = rs\begin{pmatrix}1\,1\,0\,0\\0\,0\,1\,1\end{pmatrix}, W_{24} = rs\begin{pmatrix}1\,1\,0\,1\\0\,0\,1\,1\end{pmatrix},$$

$$W_{25} = rs\begin{pmatrix}1\,0\,0\,0\\0\,0\,0\,1\end{pmatrix}, W_{26} = rs\begin{pmatrix}1\,0\,1\,0\\0\,0\,0\,1\end{pmatrix}, W_{27} = rs\begin{pmatrix}1\,1\,0\,0\\0\,0\,0\,1\end{pmatrix},$$

$$W_{28} = rs\begin{pmatrix}1\,1\,1\,0\\0\,0\,0\,1\end{pmatrix}, W_{29} = rs\begin{pmatrix}0\,1\,0\,0\\0\,0\,1\,0\end{pmatrix}, W_{30} = rs\begin{pmatrix}0\,1\,0\,0\\0\,0\,1\,1\end{pmatrix},$$

$$W_{31} = rs\begin{pmatrix}0\,1\,0\,1\\0\,0\,1\,0\end{pmatrix}, W_{32} = rs\begin{pmatrix}0\,1\,0\,1\\0\,0\,1\,1\end{pmatrix}, W_{33} = rs\begin{pmatrix}0\,1\,0\,0\\0\,0\,0\,1\end{pmatrix},$$

$$W_{34} = rs\begin{pmatrix}0\,1\,1\,0\\0\,0\,0\,1\end{pmatrix}, W_{35} = rs\begin{pmatrix}0\,0\,1\,0\\0\,0\,0\,1\end{pmatrix}.$$

Here, rs denotes the row space of the corresponding 2×4 matrix over $\mathbb{F}_q$.Under the notation of (6.4), (6.5), (6.6), the set $\mathcal{M}_{\mathbb{F}_2,4}((1,2);4)$ of maximal flag codes of type-$(1,2)$ in $\mathbb{F}_2^4$ contains exactly 328672649760 elements within the set of flag codes $\mathcal{F}_1,\ldots,\mathcal{F}_{10}$ below. Moreover, 522980 of them have been uploaded to the linkGithub

$$
\begin{aligned}
\mathcal{F}_1 &= [W_{25}, W_{17}, W_1, W_2, W_{21}, W_3, W_6, W_{13}, W_{10}, W_9, W_{30}, W_{19}, W_{23}, W_{27}, W_{22}]\,,\\
\mathcal{F}_2 &= [W_{25}, W_{17}, W_1, W_2, W_{21}, W_3, W_6, W_{13}, W_{10}, W_9, W_{30}, W_{19}, W_{28}, W_5, W_7]\,,\\
\mathcal{F}_3 &= [W_{25}, W_{17}, W_1, W_2, W_{21}, W_3, W_6, W_{13}, W_{10}, W_9, W_{30}, W_{19}, W_{28}, W_5, W_8]\,,\\
\mathcal{F}_4 &= [W_{25}, W_{17}, W_1, W_2, W_{21}, W_3, W_6, W_{13}, W_{10}, W_9, W_{30}, W_{19}, W_{28}, W_5, W_{18}]\,,\\
\mathcal{F}_5 &= [W_{25}, W_{17}, W_1, W_2, W_{21}, W_3, W_6, W_{13}, W_{10}, W_9, W_{30}, W_{19}, W_{28}, W_5, W_{22}]\,,\\
\mathcal{F}_6 &= [W_{25}, W_{17}, W_1, W_2, W_{21}, W_3, W_6, W_{13}, W_{10}, W_9, W_{30}, W_{19}, W_{28}, W_{12}, W_5]\,,\\
\mathcal{F}_7 &= [W_{25}, W_{17}, W_1, W_2, W_{21}, W_3, W_6, W_{13}, W_{10}, W_9, W_{30}, W_{19}, W_{28}, W_{12}, W_7]\,,\\
\mathcal{F}_8 &= [W_{25}, W_{17}, W_1, W_2, W_{21}, W_3, W_6, W_{13}, W_{10}, W_9, W_{30}, W_{19}, W_{28}, W_{12}, W_{18}]\,,\\
\mathcal{F}_9 &= [W_{25}, W_{17}, W_1, W_2, W_{21}, W_3, W_6, W_{13}, W_{10}, W_9, W_{30}, W_{19}, W_{28}, W_{12}, W_{22}]\,,\\
\mathcal{F}_{10} &= [W_{25}, W_{17}, W_1, W_2, W_{21}, W_3, W_6, W_{13}, W_{10}, W_9, W_{30}, W_{19}, W_{28}, W_{15}, W_5]\,.
\end{aligned} \tag{6.24}
$$

Proof. Let V be a 1-dimensional subspaces of $\mathbb{F}_2^4$. The number of 2-dimensional subspaces of $\mathbb{F}_2^4$ containing V is exactly the number of 2 by 4 reduced row echeloned matrices over $\mathbb{F}_2$ of rank 2 such that the first row is $[1000]$. Indeed if $V \subset W$ and W is a 2-dimesional subspace of $\mathbb{F}_2^4$, then considering a basis of W of the form $\{v, w\}$ with $v \in V$ shows this fact. Such reduced row echeloned matrices are

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x_1 & x_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & x_3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

with $x_1, x_2, x_3 \in \mathbb{F}_2$. Hence, the number of 2-dimensional subspaces of $\mathbb{F}_2^4$ containing V is $2^2 + 2 + 1 = 7$.

For $1 \le i \le 15$, let $J(i)$ be the subset of $\{1, 2, \ldots, 35\}$ such that $V_i \subset W_j$ if and only if $j \in J(i)$. The arguments above imply that $|J(i)| = 7$ for each $1 \le i \le 15$. Note that, if $1 \le i_1 < i_2 \le 15$, then

$$
|J(i_1) \cap J(i_2)| = 1. \tag{6.25}
$$

Let

$$
\mathcal{C} = [W_{i_1}, W_{i_2}, \ldots, W_{i_{15}}]
$$

be a maximal flag code of type-$(1,2)$ with $d = 4$ in $\mathbb{F}_2^4$ represented as in (6.21). The arguments above imply that $i_1 \in J(1), i_2 \in J(2), \ldots, i_{15} \in J(15)$. Moreover, using (6.25), we have two situations. Assume first $i_1 \in J(1)$ is chosen. The two situations depend on the following: By (6.25), $|J(1) \cap J(2)| = 1$. If $\{i_1\} = J(1) \cap J(2)$, then there are 7 choices for i_2. Using MAGMA, we search maximal flag codes of type-$(1,2)$ with $d = 4$ in $\mathbb{F}_2^4$ satisfying (6.21), (6.22), (6.23). We update the online table periodically.

$\square$

Let g be a generator of $\mathbb{F}_{2^4}^*$ with $g^4 + g + 1 = 0$. Let $\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_{35}$ be all 2-dimensional $\mathbb{F}_2$-linear subspaces of $\mathbb{F}_{2^4}$. By renumbering, we assume that $1 \in \hat{W}_1$. Let

$$\hat{W}(j) = \left\langle 1, g^j \right\rangle$$

for $1 \leq j \leq 14$. Using $g^4 + g + 1 = 0$, we obtain that

$$\begin{aligned}
\langle 1, g \rangle &= \langle 1, g^4 \rangle, \\
\langle 1, g^2 \rangle &= \langle 1, g^8 \rangle, \\
\langle 1, g^3 \rangle &= \langle 1, g^{14} \rangle, \\
\langle 1, g^5 \rangle &= \langle 1, g^{10} \rangle, \ *(\text{see Remark 9 below}) \\
\langle 1, g^6 \rangle &= \langle 1, g^{13} \rangle, \\
\langle 1, g^7 \rangle &= \langle 1, g^9 \rangle, \\
\langle 1, g^{11} \rangle &= \langle 1, g^{12} \rangle.
\end{aligned}$$

Let $\mathcal{J} = \{1, 2, 3, 6, 7, 11\}$. Note that for $j \in \mathcal{J}$, let $\mathcal{C}(j)$ be the collection given by

$$\mathcal{C}(j) = \left\{ \left[\langle 1 \rangle, \hat{W}(j) \right], \left[\langle g \rangle, g\hat{W}(j) \right], \ldots, \left[\left\langle g^{14} \right\rangle, g^{14}\hat{W}(j) \right] \right\}. \qquad (6.26)$$

Proposition 4. *Let $\mathbb{F}_q$ be a finite field with $q = 2$. Let $n = 4, T = (1,2)$ and $d = 4$. Let g be a generator of $\mathbb{F}_{2^4}^*$ and let $\mathcal{J} = \{1, 2, 3, 6, 7, 11\}$. For each $j \in \mathcal{J}$, the collection $\mathcal{C}(j)$ given in (6.26) is a maximal flag code of type-T in $\mathbb{F}_2^4$ and hence an element of $\mathcal{M}_{\mathbb{F}_2,4}((1,2);4)$. Moreover, $\mathcal{C}(j_1) \neq \mathcal{C}(j_2)$ if $j_1, j_2 \in \mathcal{J}$ and $j_1 \neq j_2$.*

Proof. Recall that $\hat{W} = \langle 1, g^j \rangle = \{0, 1, g^j, g^j + 1\}$,

$$\mathcal{C}(j) = \left\{ \left[\langle 1 \rangle , \hat{W}(j) \right], \left[\langle g \rangle , g\hat{W}(j) \right], \ldots, \left[\langle g^{14} \rangle , g^{14}\hat{W}(j) \right] \right\}.$$

As g is a generator of $\mathbb{F}_{2^4}^*$, we have $\langle g^{i_1} \rangle = \{0, g^{i_1}\} \neq \{0, g^{i_2}\} = \langle g^{i_2} \rangle$ for $1 \leq i_1 < i_2 \leq 14$. We observe that $g^{i_1}\hat{W}(j) = \{0, g^{i_1}, g^{i_1+j}, g^{i_1} + g^{i_1+j}\} \neq \{0, g^{i_2}, g^{i_2+j}, g^{i_2} + g^{i_2+j}\} = g^{i_2}\hat{W}(j)$ for $1 \leq i_1 < i_2 \leq 14$. Indeed, otherwise

$$\hat{W}(j) = g^{i_2 - i_1}\hat{W}(j).$$

Put $i = i_2 - i_1$. Note that $1 \leq i \leq 14$. We have

$$\begin{aligned} \hat{W}(j) &= \langle 1, g^j \rangle \text{ and} \\ g^i\hat{W}(j) &= \langle g^i, g^{i+j} \rangle . \end{aligned}$$

If $g^i\hat{W}(j) = \hat{W}(j)$, then $1 \in g^i\hat{W}(j)$ and $g^i \in g^i\hat{W}(j)$. These imply

$$\begin{aligned} 1 &= a.g^i + b.g^{i+j}, \\ g^j &= c.g^i + d.g^{i+j}. \end{aligned}$$

for $a, b, c, d \in \mathbb{F}_2$. Hence, we also have

$$1 = c.g^{i-j} + d.g^i = a.g^i + b.g^{i+j}. \tag{6.27}$$

Put $x = g^j$. Dividing (6.27) by g^i, we obtain

$$cx^{-1} + d = a + bx$$

and hence

$$bx^2 + (a + d)x + c = 0. \tag{6.28}$$

Using (6.28), we get a contradiction as $\mathbb{F}_2(x) = \mathbb{F}_{2^4}$ and the minimal polynomial of x over $\mathbb{F}_2$ has degree 4. Next we observe that

$$\{0, g^i\} = \langle g^i \rangle \subset g^i\hat{W}(j) = \{0, g^i, g^{i+j}, g^i + g^{i+j}\}.$$

These arguments show that $\mathcal{C}(j)$ is a maximal flag code of type-T in $\mathbb{F}_{2^4}$.

Finally, we show that $\mathcal{C}(j_1) \neq \mathcal{C}(j_2)$ if $j_1, j_2 \in \mathcal{J}$ and $j_1 \neq j_2$. Indeed
$[\langle 1 \rangle, \langle 1, g^{j_1} \rangle] \in \mathcal{C}(j_1) \setminus \mathcal{C}(j_2)$ as $[\langle 1 \rangle, \langle 1, g^{j_2} \rangle] \in \mathcal{C}(j_2)$ and
$\langle 1, g^{j_1} \rangle \neq \langle 1, g^{j_2} \rangle$.

□

Remark 9. *Note that if $j = 5$, then $g^5 \in \mathbb{F}_4$. Hence, we remove 5 in the list $\mathcal{J}$ in Proposition 4. This is another difference to the Section 6.*

Corollary 4. *Let $\mathbb{F}_q$ be a finite field with $q = 2$. Let $n = 4$, $T = (1,2)$ and $d = 4$. Let $g \in \mathbb{F}_{2^4}$ with $g^4 + g + 1 = 0$. Then the maximal flag codes of type-$(1,2)$ in $\mathbb{F}_{2^4}$ obtained by [23] correspond to the subset $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6\}$ of the ones given in (6.24).*

Remark 10. *Combining Theorem 8 and Corollary 4, we observe that we detect exactly 328672649760 maximal flag codes of type-$(1,2)$ and $d = 4$ for $q = 2$ and $n = 4$ in Theorem 8.*

CHAPTER 7

CONCLUSION

In this work, we have studied a special form of subspace codes which is helpful to increase the error-correcting capacity of the network via sending all of the previously sent ones in the new transfer. This way of using subspace codes was introduced first by Nebe et al. recently and got a lot of attention from researchers all around the world. Especially the works of Alonso et al. and Sascha Kurz gave some characterizations for some special forms. In addition, Sascha Kurz gave some results for the size of a code under some parametrizations and some upper bounds for others. He stated that his upper bounds are tight for $q = 2$. We add a new perspective to the case, we count the number of distinct maximal flag codes for some parameters and also give upper and lower bounds for any arbitrary maximal flag codes. We also extend this concept to sets and investigate the situation among a set and its subsets for various values. This extension was necessary as working with subsets lets us handle the calculations more flexibly. In this way, we find out that some of the upper bounds of Sascha Kurz are not tight for $q = 1$. This is possible with the help of modeling our flags on different types of graphs and using some combinatorial works from the literature mostly belonging to Bregman and Alon. This gives a hint that for higher q values, there might be some more interesting results.

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